Analytic methods in combinatorial number theory

("Analitiese metode in kombinatorise getalleteorie")

by

Liam Baker

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Mathematics Division,
Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, 7602 Matieland, South Africa

Supervisor: Professor Stephan Wagner

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Abstract

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L. Baker

Mathematics Division,
Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, 7602 Matieland, South Africa

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Two applications of analytic techniques to combinatorial problems with number-theoretic flavours are shown. The first is an application of the real saddle point method to derive second-order asymptotic expansions for the number of solutions to the signum equation of a general class of sequences. The second is an application of more elementary methods to yield asymptotic expansions for the number of partitions of a large integer into powers of an integer b where each part has bounded multiplicity.

Uittreksel

Analitiese metodes in kombinatoriese getalleteorie

L. Baker

Wiskunde Afdeling,
Departement van Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, 7602 Matieland, Suid Afrika

Tesis: MSc (Wisk)

Oktober 2015

Ons toon twee toepassings van analitiese tegnieke op kombinatoriese probleme met getalteoretiese geure. Die eerste is 'n toepassing van die reële saalpunt-metode wat tweede-orde asimptotiese uitbreidings vir die aantal oplossings van die 'signum' vergelyking vir 'n algemene klas van rye aflewer. Die tweede is 'n toepassing van meer elementêre metodes wat asimptotiese uitbreidings vir die aantal partisies van 'n groot heelgetal in magte van 'n heelgetal b, waar elke deel 'n begrensde meervoudigheid het, aflewer

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Contents

Preface

Analytic methods have been in use for the analysis of combinatorial and number-theoretic problems for many years, and are still in common use today. The use of analysis in number theory can, as some argue, be traced back to Riemann's seminal paper in which he defined the meromorphic continuation of the zeta function $\zeta(s)$ to the complex plane, and by deriving *analytic* properties of this functions proved the Prime Number Theorem, whereas the use of complex analytic techniques in the study of combinatorial structures was popularised by Philippe Flajolet, particularly via his book with Robert Sedgewick, aptly titled *Analytic Combinatorics*. In the latter, in most cases an expression for the generating function of a sequence is found, and then complex analytic techniques such as the saddle point method are used in order to derive asymptotic expansions of the relevant sequence.

In this thesis, two applications of analytic techniques to combinatorial structures with a number-theoretic flavour are exhibited. In the first chapter, the method used is that of the so-called *real* saddle point method, as contrasted to the *complex* saddle point method more frequently used in the field of analytic combinatorics. In the second chapter, very elementary methods are used to derive asymptotic expansions of the number of partitions of a large integer into powers of an integer b where each part has multiplicity less than d . These kinds of expansions are usually resultant from more technically advanced methods such as the aforementioned saddle point method.

A note on notation: We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, the sets of positive integers, nonnegative integers, integers, rational numbers, and real numbers respectively.

We also make frequent use of the asymptotic notation typically found in texts on analytic number theory or analytic combinatorics. In particular, for a variable x we denote by

- $O(f(x))$ a term which is bounded in magnitude by a positive multiple of $f(x)$, and

- $\Theta(f(x))$ a term which is bounded above and below by positive multiples of $f(x)$.

We also make use of the following relational symbols:

- $f(x) \ll g(x)$, being a different way of writing $f(x) = O(g(x))$, attributed to Vinogradov, and
- $f(x) \gg g(x)$, equivalent to $g(x) \ll f(x)$.

The $O(\ )$ has an advantage over the $\ll$ notation in that certain terms in complex expressions can be denoted using this notation (for instance in the expression $[\sum_{i=0}^{n} x^i + O(x^{i+1})] \times (x + x + x^2 + x^6 + O(x^{24})))$; however the $\ll$ notation has the advantage that it can be chained to form derivations such as $f(x) \ll g(x) \ll h(x) \ll i(x)$. It is for these reasons that both of these notations will be used.

There is a subtle distinction to be made between different uses of these asymptotic notations. On the one hand, the relation $f(x) \ll g(x)$ may be valid for all x, i.e. there is a constant $C > 0$ such that $f(x) \le C \cdot g(x)$ for all x, or it may be valid near to some point p[*], i.e. there is a neighbourhood U of p and a constant $C > 0$ such that $f(x) \le C \cdot g(x)$ for $x \in U$. The former, though sometimes more useful, is not always valid in circumstances in which the latter is. For instance, it is not true that $\frac{1}{x} \ll 1$ for all $x > 0$, but it is true in a neighbourhood of infinity, i.e. for x sufficiently large. To distinguish between these two cases, both of which are made use of in this thesis, in the latter case we will state that the relevant variable is *close to* some real limit point, or in the case that the limit point is infinity, that the relevant variable is *large*.

[*]which may be infinity,

Chapter 1

Erdős-Surányi and Roth-Szekeres Sequences and Trigonometric Integrals

1.1 Introduction

1.1.1 Erdős-Surányi sequences and solutions to signum equations

A sequence of positive integers $(a_n)_{n=1}^{\infty}$ is called an Erdős-Surányi sequence if every integer can be written in the form $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ for some $n \in \mathbb{N}$ and choices of signs $+$ and $-$, in infinitely many ways. Representations of this kind were first studied systematically by Erdős and Surányi [13], who provided sufficient conditions for a sequence of integers to have this property (that cover e.g. the sequence of primes).

The sequence of k-th powers, which will be of particular interest to us, was shown to be an Erdős-Surányi sequence by Mitek [16] and later independently by Bleicher [6], who also discusses the behaviour of the minimal choice of n. Drimbe [11] showed that generally, any sequence $(p(n))_{n=1}^{\infty}$ where $p(n) \in \mathbb{Q}[n]$ takes an integer value whenever $n \in \mathbb{Z}$, and $\gcd\{p(n) \mid n \in \mathbb{Z}\} = 1$, is an Erdős-Surányi -sequence, and this result was rediscovered recently by Yu [21] and also generalised further by Boulanger and Chabert [7] (to the ring of algebraic integers over a cyclotomic field), by Chen and Chen [9] (to weights other than ± 1), and again by Chen and Chen [8] (who provided a necessary and sufficient condition for arbitrary sequences of integers).

For an Erdős-Surányi sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$, the signum equation of $\mathbf{a}$ is $\pm a_i \pm a_2 \pm \cdots \pm a_n = 0$, and for a fixed $n \in \mathbb{N}$, a solution to the signum equation is a choice of $+$ and $-$ such that the equation holds. We denote the number of solutions to the signum equation of $\mathbf{a}$ by $S_{\mathbf{a}}(n)$, and more generally the number of representations of an integer k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ by $S_{\mathbf{a}}(n, k)$. In [2] it is shown that the number of solutions to the signum equation can be given by the following integral formula:

$$S_{\mathbf{a}}(n) = \frac{2^n}{2\pi} \int_0^{2\pi} \prod_{i=1}^{n} \cos(a_i t) \mathrm{d}t, \tag{1.1}$$

which follows from expanding each cosine into a sum of exponentials, multiplying out and using the fact that for $m \in \mathbb{Z}$, $\int_0^{2\pi} \exp(imt)\mathrm{d}t = 2\pi$ if $m = 0$ and 0 if $m \neq 0$. From this, it can be easily seen that the number of representations of k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ is given by $S_{\mathbf{a}}(n, k) = S_{\mathbf{a}'}(n+1)/2$ where $\mathbf{a}' = (k, a_1, a_2, \dots)$, as was shown in [3].

Andrica and Tomescu [2] conjectured that the number of solutions to the signum equation in the case $a_i = i$ is asymptotically equal to $\sqrt{6/\pi} \cdot n^{-3/2} 2^n$, which was recently proved by Sullivan [19]. The related question of representing numbers as sums of the form $\sum_{k=-n}^{n} \epsilon_k k$ with $\epsilon_k \in \{0, 1\}$ (and determining the asymptotic number of representations) was also studied in several papers, see van Lint [20], Entringer [12], Clark [10], and Louchard and Prodinger [15]. Prodinger [17] determined an asymptotic formula for the number of ways to partition the set $\{1, 2, \dots, n\}$ into two two subsets of equal cardinality and sum (note that representations of zero of the form $0 = \pm 1 \pm 2 \pm \cdots \pm n$ correspond exactly to partitions of this type, where however the cardinalities are not necessarily equal). The asymptotic behaviour of an integral similar to the one in (1.1) (but with sines rather than cosines) was studied recently in [14].

A more general conjecture in the case $a_n = n^k$ was recently formulated by Andrica and Ionaşcu [1]: namely, that

$$S_{\mathbf{a}}(n) \sim \sqrt{\frac{2(2k+1)}{\pi}} \cdot \frac{2^n}{n^{k+1/2}}.$$

The main theorem of this chapter establishes an asymptotic formula for sequences $(a_n)_{n=0}^{\infty}$ that belong to an analytic scheme due to Roth and Szekeres (see the following section). The conjecture of Andrica and Ionaşcu will be included as a special case. It will also follow that all these sequences are Erdős-Surányi sequences.

The work in this chapter is based on [5].

1.1.2 Roth-Szekeres sequences

In [18], Roth and Szekeres investigated classes of sequences $(a_n)_{n=0}^{\infty}$ satisfying the following conditions:

C1. $a_{n+1} \geq a_n$ for sufficiently large n;

C2. $s = \lim_{n\to\infty} \dfrac{\ln a_n}{\ln n}$ exists and is positive;

C3. $J_n = \inf_{(2a_n)^{-1}<t\leq 1/2} \dfrac{\sum_{i=1}^{n} \|a_i t\|^2}{\ln n} \to \infty$ as $n \to \infty$.*

For brevity, we will call such sequences Roth-Szekeres sequences. Among other results, they showed that the following classes of sequences are Roth-Szekeres sequences:

1. $a_n = p_n$, the nth prime number;
2. $a_n = f(n)$, where $f(x) \in \mathbb{Q}[x]$ takes integer values at integer places and $\gcd f(\mathbb{Z}) = \gcd\{f(n) \mid n \in \mathbb{Z}\} = 1$ (for brevity, we will call such polynomials *primitive*);
3. $a_n = f(p_n)$, where f is primitive and p_n is the nth prime number.

In particular, we see that if f is a primitive polynomial, then $(f(n))_{n=0}^{\infty}$ is both a Roth-Szekeres sequence and has been proven to be an Erdős-Surányi sequence. In fact, a corollary to the main theorem of this chapter shows that all Roth-Szekeres sequences are Erdős-Surányi sequences.

1.2 Main Theorem and applications

1.2.1 Main Theorem and sequences of applicability

For sequences that satisfy conditions C1–C3, we are able to provide an asymptotic formula for the integral in (1.1):

*Here and elsewhere for $x \in \mathbb{R}$ we denote by $\|x\|$ the distance between x and the nearest integer to x: $\|x\| = |x - \lfloor x + 1/2 \rfloor|$.

Theorem 1.1. *Let $(a_n)_{n=1}^{\infty}$ be a Roth-Szekeres sequence. Then*

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^{n} a_i^2}} - \frac{\sqrt{2\pi}\sum_{i=1}^{n} a_i^4}{8\left(\sum_{i=1}^{n} a_i^2\right)^{5/2}} + O\big(n^{-s-5/2+\epsilon}\big) \quad (1.2)$$

for any $\epsilon > 0$.

As will become clear from the proof, which makes use of the real saddle-point method, it would be possible to derive further terms of an asymptotic expansion.

Corollary 1.1.1. *If $\mathbf{a} = (a_n)_{n=0}^{\infty}$ is a Roth-Szekeres sequence, then $\mathbf{a}$ is also an Erdős-Surányi sequence.*

Proof. Let $k \in \mathbb{Z}$, and let $\mathbf{a}' = (k, a_1, a_2, \dots)$. Then for $n \in \mathbb{N}$, the number of representations of k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ is

$$\begin{aligned}
S_{\mathbf{a}}(n,k) &= \frac{2^n}{2\pi}\int_0^{2\pi} \cos(kt)\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t \\
&= \frac{2^{n+1}}{2\pi}\int_0^{\pi} \cos(kt)\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t \\
&= \frac{2^{n+1}}{2\pi}\int_0^{\pi/2} \cos(kt)\prod_{i=1}^{n}\cos(a_i t) + \cos(k(\pi-t))\prod_{i=1}^{n}\cos(a_i(\pi-t))\mathrm{d}t \\
&= \frac{2^{n+1}}{2\pi}\left(1+(-1)^{k+\sum_{i=1}^{n} a_i}\right)\int_0^{\pi/2} \cos(kt)\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t \\
&= \frac{2^n}{\sqrt{2\pi}}\left(1+(-1)^{k+\sum_{i=1}^{n} a_i}\right)\left[\frac{1}{\sqrt{\sum_{i=1}^{n} a_i^2}} - \frac{\sum_{i=1}^{n} a_i^4}{4\left(\sum_{i=1}^{n} a_i^2\right)^{5/2}} + O\big(n^{-s-5/2+\epsilon}\big)\right]
\end{aligned} \quad (1.2)$$

for all $\epsilon > 0$ by Theorem 1.1. Now since $\mathbf{a}$ is a Roth-Szekeres sequence, we have that $\sum_{i=1}^{n}\|a_i/2\|^2/\ln n \to \infty$ as $n \to \infty$ by Condition C3, and so in particular there are infinitely many $i \in \mathbb{N}$ such that a_i is odd. Hence there are infinitely many $n \in \mathbb{N}$ such that $k + \sum_{i=1}^{n} a_i$ is even. For these n, $S_{\mathbf{a}}(n,k) \to \infty$ as $n \to \infty$, and hence $\mathbf{a}$ is an Erdős-Surányi sequence. □

The following lemma expands the applicability of the above theorem:

Proposition 1.2.

1. *If $S \subset \mathbb{N}$ has the property that $\#\{k \in S \mid k \leq n\} \ll \ln n$ as $n \to \infty$ and $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is a Roth-Szekeres sequence, then the subsequence of $(a_n)_{n=1}^{\infty}$ consisting of all elements with indices* not *in S is also a Roth-Szekeres sequence for the same value of s.*

2. *If $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is a Roth-Szekeres sequence and is also a subsequence of a sequence $\mathbf{b} = (b_m)_{m=1}^{\infty}$ which satisfies Condition C1 and Condition C2 (with a possibly different value of s), then $\mathbf{b} = (b_m)_{m=1}^{\infty}$ is also a Roth-Szekeres sequence (i.e. also satisfies Condition C3).*

Proof. 1. Let $\mathbf{a}' = (a_{n_m})_{m=1}^{\infty}$ denote the subsequence of $\mathbf{a}$ with indices not in S. It is obvious that $\mathbf{a}'$ also satisfies Condition C1. Moreover, for large m we have $m = \#\{k \notin S \mid k \leq n_m\} = n_m + O(\ln n_m)$, so $\lim_{m\to\infty} \ln n_m / \ln m = \lim_{m\to\infty} n_m/m = 1$; thus

$$s = \lim_{n\to\infty} \ln a_n / \ln n = \lim_{m\to\infty} \ln a_{n_m} / \ln n_m = \lim_{m\to\infty} \ln a_{n_m} / \ln m$$

and so $\mathbf{a}'$ also satisfies Condition C2 with the same value of s. Finally,

$$\begin{aligned}\inf_{(2a_{n_m})^{-1}<t\leq 1/2} \frac{\sum_{i=1}^{m} \|a_{n_i} t\|^2}{\ln m} &\geq \inf_{(2a_{n_m})^{-1}<t\leq 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2 \; - (n_m - m)}{\ln m} \\ &= \inf_{(2a_{n_m})^{-1}<t\leq 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2 + O(\ln n_m)}{\ln m} \\ &= \inf_{(2a_{n_m})^{-1}<t\leq 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2}{\ln n_m} + O(1) \to \infty\end{aligned}$$

as $m \to \infty$, and so $\mathbf{a}'$ also satisfies Condition C3.

2. Suppose that

$$\lim_{n\to\infty} \frac{\ln a_n}{\ln n} = s_1 \quad \text{and} \quad \lim_{m\to\infty} \frac{\ln b_m}{\ln m} = s_2,$$

and that $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is the subsequence $(b_{m_n})_{n=1}^{\infty}$ of $\mathbf{b}$. It remains to show that $\mathbf{b}$ also satisfies Condition C3. Now for all $M \in \mathbb{N}$ let $n(M)$ be the greatest $n \in \mathbb{N}$ such that $m_n \leq M$. Then

$$\frac{s_2}{s_1} = \lim_{m\to\infty} \frac{\ln b_m}{\ln m} \Big/ \lim_{n\to\infty} \frac{\ln a_n}{\ln n} = \lim_{n\to\infty} \frac{\ln b_{m_n}}{\ln m_n} \Big/ \lim_{n\to\infty} \frac{\ln a_n}{\ln n} = \lim_{n\to\infty} \frac{\ln n}{\ln m_n}$$

and so

$$\lim_{M\to\infty} \frac{\ln n(M)}{\ln M} = \frac{s_2}{s_1}.$$

Finally,

$$\begin{aligned}\inf_{(2b_M)^{-1}<t\le 1/2} \frac{\sum_{m=1}^{M} \|b_m t\|^2}{\ln M} &\ge \inf_{(2b_M)^{-1}<t\le 1/2} \frac{\sum_{m_n\le M} \|b_{m_n} t\|^2}{\ln M} \\ &= \inf_{(2b_M)^{-1}<t\le 1/2} \frac{\sum_{n=1}^{n(M)} \|a_n t\|^2}{\ln n(M)} \frac{\ln n(M)}{\ln M} \\ &\sim \frac{s_2}{s_1} \inf_{(2b_M)^{-1}<t\le 1/2} \frac{\sum_{n=1}^{n(M)} \|a_n t\|^2}{\ln n(M)} \to \infty\end{aligned}$$

as $M \to \infty$ and so $\mathbf{b}$ also satisfies Condition C3.

□

1.2.2 Applications of the main theorem to more specific sequences

Polynomial-like sequences

As an application of the above theorem, consider the case when $\mathbf{a} = (a_n)_{n=0}^{\infty}$ is a Roth-Szekeres sequence and furthermore has the asymptotic expansion $a_n = \alpha n^s + \beta n^{s-1} + O(n^{s-2})$. Note that $\alpha > 0$. Then

$$\begin{aligned}a_n^2 &= \alpha^2 n^{2s} + 2\alpha\beta n^{2s-1} + O(n^{2s-2}) \\ &= \alpha^2 (2s)! \binom{n}{2s} + \alpha(2s-1)!(s(2s-1)\alpha + 2\beta) \binom{n}{2s-1} + O\big(n^{2s-2}\big)\end{aligned}$$

and

$$a_n^4 = \alpha^4 n^{4s} + O\big(n^{4s-1}\big) = \alpha^4 (4s)! \binom{n}{4s} + O\big(n^{4s-1}\big),$$

so that

$$\begin{aligned}&\sum_{i=1}^{n} a_i^2 \\ &= \alpha^2 (2s)! \binom{n+1}{2s+1} + \alpha(2s-1)!(s(2s-1)\alpha + 2\beta) \binom{n+1}{2s} + O\big(n^{2s-1}\big) \\ &= \alpha^2 (2s)! \binom{n}{2s+1} + \alpha(2s-1)!(2\alpha s + s(2s-1)\alpha + 2\beta) \binom{n}{2s} + O\big(n^{2s-1}\big) \\ &= \frac{\alpha^2}{2s+1} n^{2s+1} + \alpha\left(-\alpha s + (s+1/2)\alpha + \beta/s\right) n^{2s} + O\big(n^{2s-1}\big) \\ &= \frac{\alpha^2}{2s+1} n^{2s+1} + \alpha(\alpha/2 + \beta/s) n^{2s} + O\big(n^{2s-1}\big)\end{aligned}$$

$$\implies \frac{1}{\sqrt{\sum_{i=1}^{n} a_i^2}} = \frac{\sqrt{2s+1}}{\alpha}\left[n^{-s-1/2} - \frac{2s+1}{4}(1+2\beta/s\alpha)n^{-s-3/2}\right] + O\big(n^{-s-5/2}\big),$$

$$\text{and} \quad \sum_{i=1}^{n} a_i^4 = \alpha^4(4s)!\binom{n}{4s+1} + O\big(n^{4s}\big) = \frac{\alpha^4}{4s+1}n^{4s+1} + O\big(n^{4s}\big).$$

So by Theorem 1.1,

$$\int_0^{\pi/2}\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t = \frac{\sqrt{2\pi(2s+1)}}{2\alpha n^{s+1/2}}\left[1 - \frac{(2s+1)(s(3s+1)\alpha+(4s+1)\beta)}{2s(4s+1)\alpha n}\right] + O\big(n^{-s-5/2+\epsilon}\big) \quad \text{for any } \epsilon > 0. \tag{1.3}$$

In fact, in following the proof of Theorem 1.1 in section 1.3 for this sequence, we can see that ϵ may be set to zero.

In the special case that $\mathbf{a}$ is the polynomial sequence $a_n = n^s$, which is an Erdős-Surányi sequence as remarked earlier, we obtain the following result:

$$S_{\mathbf{a}}(n) = \left(1+(-1)^{\sum_{i=1}^{n} i}\right)\sqrt{\frac{2s+1}{2\pi}}\frac{2^n}{n^{s+1/2}}\left[1 - \frac{(2s+1)(3s+1)}{2(4s+1)n} + O\left(\frac{1}{n^2}\right)\right]$$

and so we arrive at the following asymptotic formula for the number of solutions to the signum equation of the sequence $a_n = n^k$:

$$S_{\mathbf{a}}(n) = \sqrt{\frac{2k+1}{2\pi}}\frac{2^{n+1}}{n^{k+1/2}}\left[1 - \frac{(2k+1)(3k+1)}{2(4k+1)n}\right] + O\big(2^n n^{-k-5/2}\big) \quad \text{as } n \to \infty \text{ and } n \equiv 0, 3 \bmod 4. \tag{1.4}$$

If we only have the weaker property that $a_n \sim \alpha n^s$, it still follows that

$$\int_0^{\pi/2}\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t \sim \frac{\sqrt{2\pi(2s+1)}}{2\alpha n^{s+1/2}}.$$

For example, if $\mathbf{a}$ is the sequence of square-free numbers, for which it is well known that $a_n \sim \pi^2 n/6$ (this is a Roth-Szekeres sequence, e.g. by Proposition 1.2 applied to the sequence of primes, which are all square-free), we get

$$\int_0^{\pi/2}\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t \sim \frac{3\sqrt{6}}{(\pi n)^{3/2}}.$$

Polynomials in primes

Let us also consider the case when f is a primitive polynomial, $f(n) = \alpha n^s + O(n^{s-1})$, and $a_n = f(p_n)$ where p_n is the nth prime number. We will use the following lemma:

Lemma 1.1. *If* $q(x) = \sum_{i=0}^{s} c_i x^i$ *is a polynomial with* $c_s > 0$*, then*

$$\sum_{i=1}^{n} q(p_i) \sim \frac{c_s}{s+1} n^{s+1} (\ln n)^s.$$

Proof. We imitate the proof in [4]. Let $\pi(x)$ be the prime counting function, $\mathrm{li}(x) = \int_2^x \mathrm{d}t/\ln t$ be the logarithmic integral, and define $\epsilon(x) = \pi(x) - \mathrm{li}(x)$ which is $o(t/\ln t)$ by the prime number theorem. Then we write the above sum as a Stieltjes integral, where $b < 2$:

$$\sum_{i=1}^{n} q(p_i) = \sum_{p \leq p_n} q(p) = \int_b^{p_n} q(t)\mathrm{d}\pi(t) = \int_b^{p_n} q(t)\mathrm{d}(\mathrm{li}(t) + \epsilon(t)).$$

Noting that $\mathrm{d}(\mathrm{li}(t)) = \mathrm{d}t/\ln t$, we perform integration by parts on $\int_b^{p_n} q(t)\mathrm{d}\epsilon(t)$, giving

$$\sum_{i=1}^{n} q(p_i) = \int_b^{p_n} \frac{q(t)\mathrm{d}t}{\ln t} + [q(t)\epsilon(t)]_b^{p_n} - \int_b^{p_n} \epsilon(t) q'(t)\mathrm{d}t.$$

Letting $b \to 2^-$ and noting that $\epsilon(b) \to 0$, this becomes

$$\begin{aligned}\sum_{i=1}^{n} q(p_i) &= \int_2^{p_n} \frac{q(t)\mathrm{d}t}{\ln t} + q(p_n)\epsilon(p_n) - \int_2^{p_n} \epsilon(t) q'(t)\mathrm{d}t \\ &= \sum_{i=0}^{s} c_i \left[\int_2^{p_n} \frac{t^i \mathrm{d}t}{\ln t} + p_n^i \epsilon(p_n) - \int_2^{p_n} \epsilon(t) i t^{i-1} \mathrm{d}t \right].\end{aligned}$$

In the ith summand, the first integral is an example of an exponential integral and has asymptotic expansion $\frac{p_n^{i+1}}{(i+1)\ln p_n}(1 + O(1/\ln p_n))$, whereas using the asymptotic bound on $\epsilon(t)$ it is easily seen that the other two terms are $o(p_n^{i+1}/\ln p_n)$. Hence we have the desired asymptotic expansion, using the prime number theorem result $p_n \sim n/\ln n$:

$$\sum_{i=1}^{n} q(p_i) \sim \frac{c_s}{s+1} \frac{p_n^{s+1}}{\ln p_n} \sim \frac{c_s}{s+1} n^{s+1} (\ln n)^s \qquad \square$$

Applying this lemma with $q(x) = f(x)$ (and thus $c_s = \alpha$), we get that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} f(p_n)^2 \sim \frac{\alpha^2}{2s+1} n^{2s+1} (\ln n)^{2s}.$$

Thus Theorem 1.1 gives that

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t) \mathrm{d}t \sim \frac{1}{2} \sqrt{\frac{2\pi}{\frac{\alpha^2}{2s+1} n^{2s+1} (\ln n)^{2s}}} = \frac{\sqrt{2\pi(2s+1)}}{2\alpha} \frac{1}{n^{s+1/2} (\ln n)^s},$$

$$\text{and so} \quad S_{\mathbf{a}}(n) \sim \frac{\sqrt{2s+1}}{\sqrt{2\pi}\alpha} \frac{2^{n+1}}{n^{s+1/2} (\ln n)^s} \quad \text{as } n \to \infty \text{ and } \sum_{i=1}^{n} f(p_n) \text{ is even.}$$

Specifically, if $\mathbf{a}$ is the sequence of primes, $f(n) = n$ and so $\alpha = s = 1$. Thus

$$S_{\mathbf{a}}(n) \sim \sqrt{\frac{6}{\pi}} \frac{2^n}{n^{3/2} \log n} \quad \text{as } n \to \infty \text{ for odd } n.$$

1.3 Proof of Main Theorem

To prove Theorem 1.1, we will require the following lemmas; the proofs of the first two lemmas are straightforward and thus omitted.

Lemma 1.2. *For any real number t, we have $|\cos(\pi t)| \le \exp(-\pi^2 \|t\|^2/2)$.*

Lemma 1.3. *If $b > 0$ and $\mathbf{a}$ satisfies Condition C2, then*

$$\sum_{i=1}^{n} a_i^b = O\left(n^{bs+1+\epsilon}\right) \quad \textit{and} \quad \left(\sum_{i=1}^{n} a_i^b\right)^{-1} = O\left(n^{-bs-1+\epsilon}\right) \quad \textit{for any } \epsilon > 0.$$

As a corollary, we have the following:

If $b > d > 0$ and $c \in \mathbb{R}$, then

$$\frac{\sqrt[d]{\sum_{i=1}^{n} a_i^d}}{(\ln n)^c \sqrt[b]{\sum_{i=1}^{n} a_i^b}} \to \infty \text{ as } n \to \infty, \quad \textit{and} \quad \frac{\sqrt[d]{\sum_{i=1}^{n} a_i^d}}{(\ln n)^c a_n} \to \infty \text{ as } n \to \infty.$$

Lemma 1.4. *If $\mathbf{a}$ satisfies Condition C2 and $(b_n)_{n=0}^{\infty}$ is a sequence such that $b_n > 0$ and $b_n^2 \sum_{i=1}^{n} a_i^2 / \ln n \to \infty$ as $n \to \infty$, then for any $m \ge 0$ we have*

$$\int_0^{b_n} \pi^m t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^{n} a_i^2\right) \mathrm{d}t$$
$$= 2^{(m-1)/2} \pi^{-1} \Gamma\left(\frac{m+1}{2}\right) \left(\sum_{i=1}^{n} a_i^2\right)^{-(m+1)/2} + O\left(n^{-\ell}\right) \quad \textit{for any } \ell > 0.$$

Proof.

$$\int_0^{b_n} t^m \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t = \int_0^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t \qquad (=: I_1)$$

$$- \int_{b_n}^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t. \qquad (=: I_2)$$

The first integral is

$$\begin{aligned} I_1 &= 2^{(m-1)/2}\int_0^{\infty} s^{(m-1)/2}\exp\left(-\pi^2 s\sum_{i=1}^n a_i^2\right)\mathrm{d}s \qquad \text{for } s=\frac{t^2}{2} \\ &= 2^{(m-1)/2}\left(\pi^2\sum_{i=1}^n a_i^2\right)^{-(m+1)/2}\int_0^{\infty} q^{(m-1)/2}e^{-q}\mathrm{d}q \quad \text{for } q=\pi^2\sum_{i=1}^n a_i^2 s \\ &= 2^{(m-1)/2}\pi^{-m-1}\Gamma\left(\frac{m+1}{2}\right)\left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2}, \end{aligned}$$

and by a similar procedure the second integral can be written as follows:

$$\begin{aligned} &\int_{b_n}^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=}^n a_i^2\right)\mathrm{d}t \\ &\qquad = 2^{(m-1)/2}\pi^{-m-1}\left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2}\int_{x_n}^{\infty} q^{(m-1)/2}e^{-q}\mathrm{d}q \end{aligned}$$

where $x_n = \pi^2 b_n^2 \sum_{i=1}^n a_i^2/2 \to \infty$ as $n\to\infty$. Now for q sufficiently large, $q^{(m-1)/2} \le e^{q/2}$, so that for n sufficiently large,

$$\int_{x_n}^{\infty} q^{(m-1)/2}e^{-q}\mathrm{d}q \le \int_{x_n}^{\infty} e^{-q/2}\mathrm{d}q = 2e^{-x_n/2} = O\big(n^{-\ell}\big) \quad \text{for any } \ell > 0.$$

Finally, $a_n\to\infty$ and so $\left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2} \to 0$ as $n\to\infty$, so the estimate is proved. □

Now we are ready to prove Theorem 1.1:

Proof of Theorem 1.1. We assume throughout that n is large, and thus that a_n is large positive and not less than a_i for $i < n$. This proof is an implementation of the real saddle point method, which is used often in analytic combinatorics and number theory. In this method, which is usually used to approximate an integral, one shows that over a large part of the region of integration (called the 'tail' or 'tails' of the integral) the integrand is negligibly small. To this end, we rewrite the integral in Equation 1.2 as follows:

$$\begin{aligned}\int_0^{\pi/2}\prod_{i=1}^{n}\cos(a_i t)\mathrm{d}t = \pi\int_0^{1/(2a_n)}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t \qquad & (=: I_1)\\ +\,\pi\int_{1/(2a_n)}^{1/2}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t. \qquad & (=: I_2)\end{aligned}$$

Here the second integral, I_2, forms part of the tail of the integral and can be estimated as follows using Condition C3 and Lemma 1.2:

$$\begin{aligned}\left|\int_{1/(2a_n)}^{1/2}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t\right| &\le \int_{1/(2a_n)}^{1/2}\prod_{i=1}^{n}|\cos(a_i\pi t)|\mathrm{d}t\\ \le \int_{1/(2a_n)}^{1/2}\prod_{i=1}^{n}\exp\left(-\frac{\pi^2}{2}\|a_i t\|^2\right)\mathrm{d}t &= \int_{1/(2a_n)}^{1/2}\exp\left(-\frac{\pi^2}{2}\sum_{i=1}^{n}\|a_i t\|^2\right)\mathrm{d}t\\ \le \int_{1/(2a_n)}^{1/2}\exp\left(-\frac{\pi^2}{2}J_n\ln n\right)\mathrm{d}t &= \left[\frac{1}{2}-\frac{1}{2a_n}\right]\exp\left(-\frac{\pi^2}{2}J_n\ln n\right)\\ &< \frac{1}{2}n^{-\pi^2 J_n/2}.\end{aligned}$$

Since $J_n \to \infty$ as $n \to \infty$, it follows that

$$I_2 = \pi\int_{1/2a_n}^{1/2}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t = O(n^{-\ell}) \quad \text{for any } \ell > 0. \tag{1.5}$$

We now split up the first integral, I_1, again:

$$\begin{aligned}\pi\int_0^{1/2a_n}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t = \pi\int_0^{b_n}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t \qquad & (=: I_3)\\ +\,\pi\int_{b_n}^{1/2a_n}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t \qquad & (=: I_4)\end{aligned}$$

where $b_n \in (0, 1/2a_n)$ will be chosen later. I_4 can be estimated as follows:

$$\begin{aligned}\left|\int_{b_n}^{1/2a_n} \prod_{i=1}^{n} \cos(a_i\pi t)\mathrm{d}t\right| &\leq \int_{b_n}^{1/2a_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^{n} a_i^2\right)\mathrm{d}t \\ &< \int_{b_n}^{1/2a_n} \exp\left(-\frac{\pi^2 b_n^2}{2}\sum_{i=1}^{n} a_i^2\right)\mathrm{d}t \\ &< \int_{0}^{1/2a_n} \exp\left(-\frac{\pi^2 b_n^2}{2}\sum_{i=1}^{n} a_i^2\right)\mathrm{d}t \\ &= \frac{1}{2a_n}\exp\left(-\frac{\pi^2 b_n^2}{2}\sum_{i=1}^{n} a_i^2\right).\end{aligned}$$

We then have the following estimate for I_4 which is the rest of the tail:

$$I_4 = O(n^{-\ell}) \text{ for any } \ell > 0, \quad \text{provided that } b_n^2 \sum_{i=1}^{n} a_i^2/\ln n \xrightarrow{n\to\infty} \infty. \tag{1.6}$$

The second part of the saddle point method is to approximate the integrand by a more easily integrable function in the region excluding the tail. This region is usually a neighbourhood of a certain point, which is called the saddle point. In this case, the saddle point is $t = 0$, and we now approximate our integrand for t close to zero to obtain an approximation to our remaining integral, I_3:

Now $g(t) := \ln\cos t$ can be expanded in a Taylor series valid for $|t| < \pi/2$, and hence for any $\beta \in (0, \pi/2)$ we have a constant C_β such that for $|t| \leq \beta$,

$$\begin{aligned}&|h(t)| \\ &= \left|\ln\cos t + \frac{t^2}{2} + \frac{t^4}{12}\right| \\ &= \left|g(t) - g(0) - g^{(1)}(0)t - \frac{g^{(2)}(0)}{2}t^2 - \frac{g^{(3)}(0)}{6}t^3 - \frac{g^{(4)}(0)}{24}t^4 - \frac{g^{(5)}(0)}{120}t^5\right| \\ &\leq C_\beta t^6\end{aligned}$$

where $h(t) = \ln\cos t + t^2/2 + t^4/12$. Then for $|t| \leq \beta$,

$$\ln\cos t = -\frac{t^2}{2} - \frac{t^4}{12} + h(t) \quad \Rightarrow \quad \cos t = \exp\left(-\frac{t^2}{2} - \frac{t^4}{12}\right)e^{h(t)}.$$

In particular, for $\beta = \pi a_n b_n$ we get that for $|t| \leq b_n$ we have

$$\prod_{i=1}^{n} \cos(a_i\pi t) = \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^{n} a_i^2 - \frac{\pi^4 t^4}{12}\sum_{i=1}^{n} a_i^4\right)\exp\left(\sum_{i=1}^{n} h(a_i\pi t)\right) \quad \text{so that}$$

$$\left|\frac{\prod_{i=1}^{n}\cos(a_i\pi t)}{\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2-\frac{\pi^4t^4}{12}\sum_{i=1}^{n}a_i^4\right)}-1\right|=\left|\exp\left(\sum_{i=1}^{n}h(a_i\pi t)\right)-1\right|.$$

Since the exponential function is increasing, we have that

$$\exp\left(-C_\beta\pi^6t^6\sum_{i=1}^{n}a_i^6\right)\leq\exp\left(\sum_{i=1}^{n}h(a_i\pi t)\right)\leq\exp\left(C_\beta\pi^6t^6\sum_{i=1}^{n}a_i^6\right),$$

$$\text{and so}\quad\left|\exp\left(\sum_{i=1}^{n}h(a_i\pi t)\right)-1\right|\leq\exp\left(C_\beta\pi^6t^6\sum_{i=1}^{n}a_i^6\right)-1$$

since the exponential function is convex.

Now if $b_n^6\sum_{i=1}^{n}a_i^6\to 0$ as $n\to\infty$, we will have that since $\exp x$ has a Taylor series expansion valid for all finite values, $e^{C_\beta\pi^6x}-1=O(x)$ for $|x|<\max_{n\in\mathbb{N}}\{b_n^6\sum_{i=1}^{n}a_i^6\}$. Then for $|t|\leq b_n$,

$$\begin{aligned}&\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2-\frac{\pi^4t^4}{12}\sum_{i=1}^{n}a_i^4\right)\left|\exp\left(\sum_{i=1}^{n}h(a_i\pi t)\right)-1\right|\\&\leq\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2\right)\left|\exp\left(\sum_{i=1}^{n}h(a_i\pi t)\right)-1\right|\\&=\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2\right)\cdot O\left(t^6\sum_{i=1}^{n}a_i^6\right).\end{aligned}$$

Thus we have that

$$\begin{aligned}&\int_0^{b_n}\prod_{i=1}^{n}\cos(a_i\pi t)-\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2-\frac{\pi^4t^4}{12}\sum_{i=1}^{n}a_i^4\right)\mathrm{d}t\\&\ll\sum_{i=1}^{n}a_i^6\int_0^{b_n}t^6\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2\right)\mathrm{d}t\\&=O\left(\sum_{i=1}^{n}a_i^6\int_0^{\infty}t^6\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2\right)\mathrm{d}t\right)=O\left(\frac{\sum_{i=1}^{n}a_i^6}{(\sum_{i=1}^{n}a_i^2)^{7/2}}\right)^\dagger,\end{aligned}$$

and so

$$\int_0^{b_n}\prod_{i=1}^{n}\cos(a_i\pi t)\mathrm{d}t=\int_0^{b_n}\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^{n}a_i^2-\frac{\pi^4t^4}{12}\sum_{i=1}^{n}a_i^4\right)\mathrm{d}t+O\left(\frac{\sum_{i=1}^{n}a_i^6}{(\sum_{i=1}^{n}a_i^2)^{7/2}}\right)$$

† from the proof of Lemma 1.4

provided that $b_n^6 \sum_{i=1}^n a_i^6 \to 0$ as $n \to \infty$. Note that by Lemma 1.3,

$$\sum_{i=1}^n a_i^6 \bigg/ \left(\sum_{i=1}^n a_i^2\right)^{7/2} = O\left(n^{-s-5/2+\epsilon}\right) \quad \text{for any } \epsilon > 0.$$

Hence we have the following estimate:

$$I_3 = \pi \int_0^{b_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2 - \frac{\pi^4 t^4}{12}\sum_{i=1}^n a_i^4\right) \mathrm{d}t \\ + O\left(n^{-s-5/2+\epsilon}\right) \quad \text{for any} \quad \epsilon > 0, \quad (1.7)$$

provided that $b_n^6 \sum_{i=1}^n a_i^6 \xrightarrow{n\to\infty} 0$.

Now using the Taylor expansion for the exponential function, we have that

$$\exp\left(-\frac{\pi^4 t^4}{12}\sum_{i=1}^n a_i^4\right) = 1 - \frac{\pi^4 t^4}{12}\sum_{i=1}^n a_i^4 + O\left(t^8\left(\sum_{i=1}^n a_i^4\right)^2\right) \text{ for } 0 < t < b_n, \tag{1.8}$$

provided that $b_n^4 \sum_{i=1}^n a_i^4 \to 0$ as $n \to \infty$. Then by Lemma 1.4,

$$\begin{aligned} &\int_0^{b_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2 - \frac{\pi^4 t^4}{12}\sum_{i=1}^n a_i^4\right) \mathrm{d}t \\ =& \int_0^{b_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t - \sum_{i=1}^n a_i^4 \int_0^{b_n} \frac{\pi^4 t^4}{12} \exp\left(-\frac{-\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t \\ &+ O\left(\left(\sum_{i=1}^n a_i^4\right)^2 \int_0^{b_n} t^8 \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right) \mathrm{d}t\right) \\ =& \frac{1}{\sqrt{2\pi}}\Gamma(1/2)\left(\sum_{i=1}^n a_i^2\right)^{-1/2} - \frac{2^{3/2}}{12\pi}\Gamma(5/2)\sum_{i=1}^n a_i^4\left(\sum_{i=1}^n a_i^2\right)^{-5/2} \\ &+ O\left(\left(\sum_{i=1}^n a_i^4\right)^2\left(\sum_{i=1}^n a_i^2\right)^{-9/2}\right) + O\left(n^{-\ell}\right) \quad \text{for any } \ell > 0. \end{aligned}$$

Now by Lemma 1.3, we have $\left(\sum_{i=1}^n a_i^4\right)^2 \left(\sum_{i=1}^n a_i^2\right)^{-9/2} = O\left(n^{-s-5/2+\epsilon}\right)$ for any $\epsilon > 0$. Hence

$$\int_0^{b_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2 - \frac{\pi^4 t^4}{12}\sum_{i=1}^n a_i^4\right) \mathrm{d}t$$
$$= \frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^n a_i^2\right)^{-1/2} - \frac{1}{4\sqrt{2\pi}}\sum_{i=1}^n a_i^4 \left(\sum_{i=1}^n a_i^2\right)^{-5/2} + O\big(n^{-s-5/2+\epsilon}\big)$$

for any $\epsilon > 0$. This combined with Equation 1.7 shows that

$$I_3 = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^n a_i^2}} - \frac{\sqrt{2\pi}}{8}\frac{\sum_{i=1}^n a_i^4}{\left(\sum_{i=1}^n a_i^2\right)^{5/2}} + O\big(n^{-s-5/2+\epsilon}\big) \quad \text{for any } \epsilon > 0. \tag{1.9}$$

Finally, combining Equation 1.5, Equation 1.6 and Equation 1.9 we arrive at the following second-order approximation for our initial integral:

$$\int_{-\pi/2}^{\pi/2}\prod_{i=1}^n \cos(a_i t)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^n a_i^2}} - \frac{\sqrt{2\pi}\sum_{i=1}^n a_i^4}{8\left(\sum_{i=1}^n a_i^2\right)^{5/2}} + O\big(n^{-s-5/2+\epsilon}\big) \tag{1.10}$$

for any $\epsilon > 0$.

The only issue which yet remains is the existence of a sequence $(b_n)_{n=1}^{\infty}$ satisfying the conditions imposed on it in Lemma 1.4, Equation 1.6, Equation 1.7 and Equation 1.8. Using Lemma 1.3, it is easy to see that $b_n = n^{-s-1/3}$ satisfies these conditions, and hence our proof is complete.

□

Bibliography

[1] Dorin Andrica and Eugen J. Ionascu. "Some Unexpected Connections Between Analysis and Combinatorics". In: Mathematics Without Boundaries. Springer, 2014, pp. 1–19.

[2] Dorin Andrica and Ioan Tomescu. "On an Integer Sequence Related to a Product of Trigonometric Functions, and its Combinatorial Relevance". In: *Journal of Integer Sequences* 5.2 (2002), p. 3.

[3] Dorin Andrica and Daniel Văcăreţu. "Representation theorems and almost unimodal sequences". In: *Studia. Universitatis Babeş-Bolyai. Mathematica* 51.4 (2006), pp. 23–33.

[4] Eric Bach. "Algorithmic Number Theory: Efficient Algorithms". In: vol. 1. MIT press, 1996, p. 29.

[5] Liam Baker and Stephan Wagner. "Erdős-Surányi sequences and trigonometric integrals". In: *Preprint* (2015). URL: `http://arxiv.org/abs/1506.04555`.

[6] Michael N. Bleicher. "On Prielipp's problem on signed sums of kth powers". In: *Journal of Number Theory* 56.1 (1996), pp. 36–51.

[7] Jacques Boulanger and Jean-Luc Chabert. "On the representation of integers as linear combinations of consecutive values of a polynomial". In: *Transactions of the American Mathematical Society* 356.12 (2004), pp. 5071–5088.

[8] Feng-Juan Chen and Yong-Gao Chen. "Signed sums of terms of a sequence". In: *Proceedings of the American Mathematical Society* 141.4 (2013), pp. 1105–1111.

[9] Feng-Juan Chen and Yong-Gao Chen. "Weighted sums of consecutive values of a polynomial". In: *Journal of Number Theory* 132.12 (2012), pp. 2725–2735.

[10] Lane Clark. "On the representation of m as $\sum_{k=-n}^{n} \varepsilon_k k$". In: *International Journal of Mathematics and Mathematical Sciences* 23 (2000), pp. 77–80.

[11] Mihai Onucu Drimbe. "Generalization of representation theorem of Erdős and Surányi". In: *Roczniki Polskiego Towarzystwa Matematycznego. Seria I. Commentationes Mathematicae. Prace Matematyczne* 27.2 (1988), pp. 233–235.

[12] R. C. Entringer. "Representation of m as $\sum_{k=-n}^{n} \varepsilon_k k$". In: *Canadian Mathematical Bulletin* 11 (1968), pp. 289–293.

[13] Pál Erdös and János Surányi. "Über ein Problem aus der additiven Zahlentheorie". In: *Matematikai Lapok. Bolyai János Matematikai Társulat* 10 (1959), pp. 284–290.

[14] Jeffrey Gaither et al. "Resolution of T. Ward's question and the Israel-Finch conjecture: precise analysis of an integer sequence arising in dynamics". In: *Combinatorics, Probability and Computing* 24.1 (2015), pp. 195–215.

[15] Guy Louchard and Helmut Prodinger. "Representations of numbers as $\sum_{k=-n}^{n} \varepsilon_k k$: A saddle point approach". In: *Infinity in Logic and Computation*. Vol. 5489. Lecture Notes in Computer Science. Springer, 2010, pp. 87–96.

[16] Jerzy Mitek. "Generalization of a theorem of Erdős and Surányi". In: *Roczniki Polskiego Towarzystwa Matematycznego. Seria I. Commentationes Mathematicae. Prace Matematyczne* 21.1 (1980), pp. 173–175.

[17] Helmut Prodinger. "On the number of partitions of $\{1, ..., N\}$ into two sets of equal cardinalities and equal sums". In: *Canadian Mathematical Bulletin* 25 (1982), pp. 238–241.

[18] Klaus Friedrich Roth and George Szekeres. "Some asymptotic formulae in the theory of partitions". In: *The Quarterly Journal of Mathematics* 5.1 (1954), pp. 241–259.

[19] Blair D. Sullivan. "On a Conjecture of Andrica and Tomescu". In: *Journal of Integer Sequences* 16.2 (2013), p. 3.

[20] Jacobus Hendricus van Lint. "Representation of 0 as $\sum_{k=-N}^{N} \varepsilon_k k$". In: *Proceedings of the American Mathematical Society* 18 (1967), pp. 182–184.

[21] Hong Bing Yu. "Signed sums of polynomial values". In: *Proceedings of the American Mathematical Society* 130.6 (2002), pp. 1623–1627.

Chapter 2

The Combinatorics of Redundant Single-base Numeration Systems

2.1 Introduction

We define the sequence $a_{d,m}$ to be the number of ways in which an integer m can be written in a base b expansion where the 'digits' are allowed to be any nonnegative integers less than a finite positive integer d, i.e. the number of solutions to

$$m = \sum_{n=0}^{\infty} e_n b^n \quad \text{where } 0 \leq e_n < d \text{ for each } n \in \mathbb{N}_0. \tag{2.1}$$

Here b is a positive integer greater than 1. An equivalent definition of $a_{d,m}$ is as the number of partitions of m where each part is a power of b and each part occurs less than d times. Hence or otherwise, it is easy to see that the generating function for the sequence $a_{d,m}$ is

$$\begin{aligned} G_d(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \left(1 + x + \cdots + x^{d-1}\right)\left(1 + x^b + \cdots + x^{b(d-1)}\right)\left(1 + x^{b^2} + \cdots + x^{b^2(d-1)}\right)\cdots \\ &= \prod_{n=0}^{\infty} \frac{1 - x^{b^n d}}{1 - x^{b^n}}, \end{aligned} \tag{2.2}$$

and satisfies the recurrence

$$G_d(x) = \frac{1 - x^d}{1 - x}\, G_d(x^b). \tag{2.3}$$

Unless d is explicitly referenced, the notation $a_{d,m}$ will be shortened to a_m.

The arithmetic properties of this sequence a_m have been studied by various authors since Churchhouse's conjecture in [6], which was later proved and extended by Rødseth [20], Gupta [11], Andrews [4], Dirdal [9, 8, 7] and in more recent years by Hirschhorn and Sellers [12], Rødseth and Sellers [22, 21] and Anders, Dennison, Lansing and Reznick [3].

We are, however, interested in the asymptotic growth and behaviour of these sequences, which for infinite d and $b = 2$ were studied first by Euler and Tanturri, and then later by Knuth [13] and also Reznick [23]. The asymptotic properties of the case with general b and $d = \infty$ were first studied by Mahler [15] and later by de Bruijn [5] and Pennington [17], who found that

$$\begin{aligned}\log a_{\infty,bm} = {} & \frac{1}{2\log b}\left(\log\frac{m}{\log m}\right)^2 + \left(\frac{1}{2} + \frac{1}{\log b} + \frac{\log\log b}{\log b}\right)\log m \\ & - \left(1 + \frac{\log\log b}{\log b}\right)\log\log m + \psi\left(\frac{\log m - \log\log m}{\log b}\right) + o(1),\end{aligned}$$

where ψ is a periodic function of period 1, and also found an asymptotic formula for the summatory function $\sum_{m'<m} a_{\infty,m'}$. In fact, Pennington's method works for a larger class of examples than consecutive powers of b.

Now if an integer m can be written in the form of Equation 2.1, then we must have that $e_0 \equiv m \mod b$, and moreover for each $e_0 \in [0, d)$ with $e_0 \equiv m \mod b$, $\sum_{n=0}^{\infty} e_{n+1} b^n$ is a partition of $(m - e_0)/b$ into powers of b. Hence we obtain the following recurrence relation for $a_{d,m}$:

$$a_m = \sum_{\substack{0\le r<d \\ r\equiv m \mod b}} a_{\frac{m-r}{b}}. \tag{2.4}$$

In particular, if $d = bf + r$ where $f \in \mathbb{Z}$ and $0 \le r < b$, then

$$a_{bm+r-1} = a_{bm+r-2} = \cdots = a_{bm} = a_m + a_{m-1} + \cdots + a_{m-\lceil d/b\rceil+1}$$

and

$$a_{bm-1} = a_{bm-2} = \cdots = a_{bm-b+r} = a_{m-1} + a_{m-2} + \cdots + \underbrace{a_{m-\lfloor d/b\rfloor}}_{=a_{m-f}}. \tag{2.5}$$

From this it follows that a_m is a b-regular sequence in m in the sense of Allouche and Shallit [2, 1]. This does not, unfortunately, in itself give much information regarding the *asymptotics* of a_m. These asymptotics were recently studied by Protasov [18, 19], who showed that $p_d = \liminf_{m\to\infty} \frac{\log a_m}{\log m}$ and $q_d = \limsup_{m\to\infty} \frac{\log a_m}{\log m}$ both exist, calculated them explicitly for some pairs (b, d) and wrote formulas for them in terms of the joint spectral radius of a finite family of matrices for others, and established bounds for these 'growth exponents' in his earlier paper, and in his later paper showed that for $b = 2$, these growth exponents are equal when d is even and unequal when d is odd, and also showed that the limits $\liminf_{m\to\infty} a_m/m^{p_d}$ and $\limsup_{m\to\infty} a_m/m^{q_d}$ both exist and are positive when d is even, but are only equal when d is a power of 2. Much more recently, in [10], Feng, Liardet and Thomas give first-order asymptotic expansions for partitions into classes of parts which include that of consecutive powers of b, and in [14] the asymptotic behaviour of the number of partitions into parts each of which is a product of the form $b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_m^{\alpha_m}$ where $b_1, b_2, \ldots, b_m$ are fixed positive integers was studied in the case that $m > 2$. Unfortunately their method, which was based on saddle-point analysis of the associated generating functions, does not work as well in the case of a single base, or $m = 1$. In this thesis we generalise some of Protasov's results to the case of general b, filling in part of the gap in [14], and give some new results, the type of which are not found in Protasov's work or in [10] or in [14].

2.2 Growth of the a_m

Since m has at least as many representations wherein the 'digits' can be at most d as those in which each 'digit' must be less than d, it follows that the double sequence $a_{d,m}$ satisfies the following inequality:

$$a_{d+1,m} \geq a_{d,m},$$

and so $a_{d,m}$ is nondecreasing in d.

Now when $d = b^l$ for some $l \in \mathbb{N}$, the generating function $G_d(x)$ is

$$G_d(x) = \frac{1}{\prod_{n=0}^{l-1} 1 - x^{b^n}} = \frac{1}{(1-x)(1-x^b)\cdots(1-x^{b^{l-1}})}.$$

This rational function may then be written as a sum of partial fractions:

$$G_d(x) = \sum_{n=0}^{l-1} \sum_{\substack{\zeta^{b^n}=1 \\ \zeta^{b^{n-1}}\neq 1}} p_n \left(\frac{1}{1-\zeta x}\right),$$

where the second sum above is over all b^nth roots of unity ζ which are not b^rth roots of unity for any $r < n$, and each p_n is a polynomial of degree $l - n$. Then using the binomial theorem on each partial fraction, we see that

$$a_m = [x^m]\, G_d(x) = Cn^{l-1} + O\left(n^{l-2}\right)$$

for some $C > 0$.*

More generally, the question arises of for which d the value

$$\lim_{m\to\infty} \frac{\log a_{d,m}}{\log m}$$

exists. The above shows that it does exist for $d = b^l$ when $l \in \mathbb{N}$, and there it is equal to $l - 1 = \log_b b^l - 1$. Using this and the fact that $a_{d,m}$ is nondecreasing in d, we have that

$$p_d = \liminf_{m\to\infty} \frac{\log a_{d,m}}{\log m} \quad \text{and} \quad q_d = \limsup_{m\to\infty} \frac{\log a_{d,m}}{\log m} \tag{2.6}$$

both exist for each $d \geq b$; of course, when they are equal, the above limit exists and is equal to their common value. When $1 < d < b$, it is easy to see that each a_m is either zero or one, and that each possibility occurs infinitely often; hence $p_d = -\infty$ and $q_d = 0$ there.† When $d = 1$, $a_0 = 1$ and $a_m = 0$ for $m > 0$, so $p_d = q_d = -\infty$.

Since $a_{d,m}$ is nondecreasing in d, we have that for $d_1 < d_2$, $p_{d_1} \leq p_{d_2}$ and $q_{d_1} \leq q_{d_2}$. We will prove that for $d \geq b$, $p_d \leq \log_b d - 1 \leq q_d$, from which it follows that if $\lim_{m\to\infty} \log a_m / \log m$ exists, it is equal to $\log_b d - 1$; however before doing so we will require the following result:

Proposition 2.1. *For $d_1, d_2 \in \mathbb{N}$,*

$$a_{d_1 \times d_2, m} = \sum_{k + d_1 l = m} a_{d_1,k} a_{d_2,l}.$$

Proof. We prove the above result by establishing a combinatorial bijection. Denote by $[d]$ the set $\{0, 1, \ldots, d-1\}$ for each $d \in \mathbb{N}$, and consider a representation of m as in Equation 2.1 for $d = d_1 d_2$:

$$m = \sum_{n=0}^{\infty} e_n b^n \quad \text{where each } e_n \in [d_1 d_2].$$

*Here and elsewhere, the notation $[x^m]\, f(x)$ denotes the coefficient of x^m in the generating function f.

†We use the convention that for $x > 1$, $\log_x 0 = -\infty$ and $x^{-\infty} = 0$.

Now each e_n can be uniquely written as $f_n+d_1g_n$ where $f_n \in [d_1]$ and $g_n \in [d_2]$. So the above representation splits as follows:

$$m = \sum_{n=0}^{\infty}(f_n + d_1 g_n)b^n = \sum_{n=0}^{\infty} f_n b^n + d_1 \sum_{n=0}^{\infty} g_n b^n = k + d_1 l,$$

where each choice of the e_n yields a different pair of 'base b' expansions of $k = \sum_{n=0}^{\infty} f_n b^n$ and $l = \sum_{n=0}^{\infty} g_n b^n$. Moreover, given any value of k and l such that $k + d_1 l = m$, each pair of representations of k as in Equation 2.1 with $d = d_1$ and of l with $d = d_2$ yields a different representation of m with $d = d_1 d_2$, by defining $e_n = f_n + d_1 g_n$ for each n. Hence we have a bijection between the set of representations of m and the set of sets of pairs of representations of k and of l for k and l which satisfy $k + d_1 l = m$. In particular,

$$a_{d_1 d_2, m} = \sum_{k+d_2 l = m} a_{d_1,k} a_{d_2,l}$$

as desired. □

We now use the above result to prove two inequalities relating the p_d and the q_d for different d:

Theorem 2.2. *For $d_1, d_2 \geq b$,*

$$p_{d_1 d_2} \geq p_{d_1} + p_{d_2} + 1 \quad \textit{and} \quad q_{d_1 d_2} \leq q_{d_1} + q_{d_2} + 1.$$

Proof.

$$\begin{aligned} p_d &= \liminf_{m\to\infty} \frac{\log a_m}{\log m} = \sup\{p \mid \log_m a_{d,m} \geq p \text{ for large } m\} \\ &= \sup\{p \mid a_{d,m} \geq m^p \text{ for large } m\} \\ &= \sup\{p \mid a_{d,m} \gg m^p\} \end{aligned}$$

and similarly

$$q_d = \inf\{q \mid a_{d,m} \ll m^q \text{ for large } m\}.^{\ddagger}$$

To prove the first desired inequality, we show that if $a_{d_1,m} \gg m^{p_1}$ and $a_{d_2,m} \gg m^{p_2}$ for large m, then $a_{d_1 d_2,m} \gg m^{p_1+p_2+1}$ for large m; for then $p_{d_1 d_2} \geq p_1 + p_2 + 1$ being true for all $p_1 < p_{d_1}$ and $p_2 < p_{d_2}$ implies that $p_{d_1 d_2} \geq p_1 + p_2 + 1$ as desired.

‡Actually, in [18] Protasov showed more; namely that $\limsup_{m\to\infty} \frac{a_{d,m}}{m^{q_d}} \in (0,\infty)$ and $\liminf_{m\to\infty} \frac{a_{d,m}}{m^{p_d}} \in (0,\infty)$.

So suppose that $a_{d_1,m} \gg m^{p_1}$ and $a_{d_2,m} \gg m^{p_2}$ for large m. Then by Proposition 2.1,

$$\begin{aligned} a_{d_2 d_2,m} &= \sum_{k+d_1 l=m} a_{d_1,k} a_{d_2,l} \geq \sum_{\substack{k+d_1 l=m \\ k\geq m/3 \\ d_1 l \geq m/3}} a_{d_1,k} a_{d_2,l} \\ &\gg \sum_{\substack{k+d_1 l=m \\ k\geq m/3 \\ d_1 l \geq m/3}} k^{p_1} l^{p_2} \\ &\geq \sum_{\substack{k+d_1 l=m \\ k\geq m/3 \\ d_1 l \geq m/3}} \left(\frac{m}{3}\right)^{p_1} \left(\frac{m}{3d_1}\right)^{p_2} \\ &\gg m \cdot m^{p_1} \cdot m^{p_2} = m^{p_1+p_2+1} \end{aligned}$$

and so the first inequality is proved.

We prove the second inequality similarly: suppose that $a_{d_1,m} \ll m^{q_1}$ and $a_{d_2,m} \ll m^{q_2}$ for large m, and note then that the ineffective inequalities $a_{d_1,m} \ll (m+1)^{q_1}$ and $a_{d_2,m} \ll (m+1)^{q_2}$ are valid for all $m \geq 0$. Thus for large m, by Proposition 2.1

$$\begin{aligned} a_{d_1 d_2,m} &= \sum_{\substack{k+d_1 l=m \\ k,l\geq 0}} a_{d_1,k} a_{d_2,l} \\ &\ll \sum_{\substack{k+d_1 l=m \\ k,l\geq 0}} (k+1)^{q_1} (l+1)^{q_2} \\ &\leq \sum_{\substack{k+d_1 l=m \\ k,l\geq 0}} (m+1)^{q_1} (m+1)^{q_2} \leq (m+1)^{q_1+q_2+1} \\ &\ll m^{q_1+q_2+1}. \end{aligned}$$

and so $q_{d_1 d_2} \leq q_1 + q_2 + 1$. Since the inequalities $a_{d_1,m} \ll m^{q_1}$ and $a_{d_2,m} \ll m^{q_2}$ are true for all $q_1 > q_{d_1}$ and $q_2 > q_{d_2}$, this implies that $q_{d_1 d_2} \leq q_{d_1} + q_{d_2} + 1$ as desired. □

Now Figure 2.1 and Figure 2.2 show plots of the quantity $\log a_{d,n} / \log n$ for $b = 3$ and d between 4 and 11, and they suggest that p_d and q_d are equal when d is a multiple of b. The following proposition proves this hypothesis true:

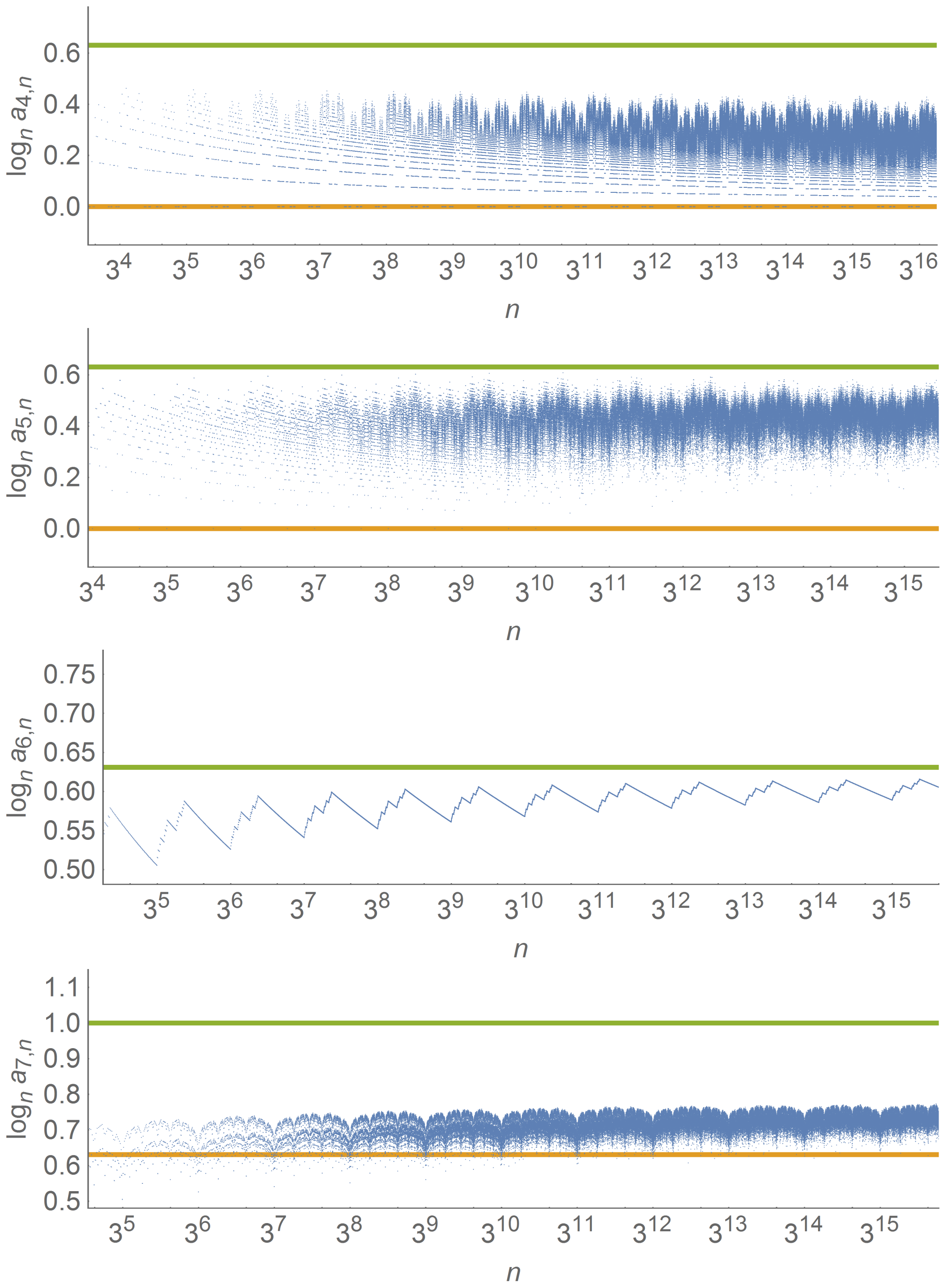

Figure 2.1: Plots of $\log a_{d,n}/\log n$ versus n (n being on a logarithmic scale) for $b = 3$ and $d = 4, 5, 6$ and 7. The orange and green horizontal lines are the values of $\log_b \lfloor c/b \rfloor$ and $\log_b \lceil c/b \rceil$, respectively.

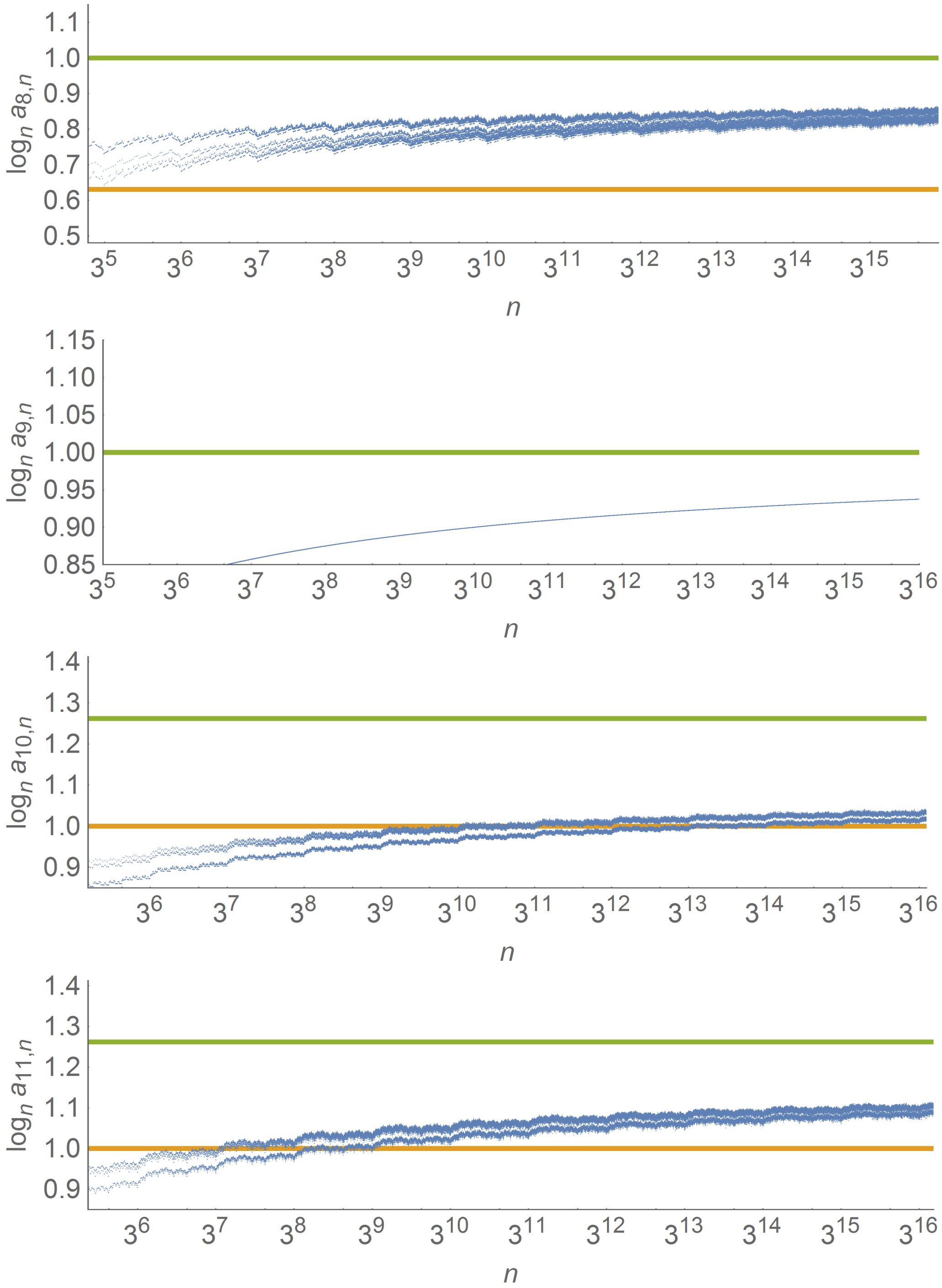

Figure 2.2: Plots of $\log a_{d,n}/\log n$ versus n (n being on a logarithmic scale) for $b = 3$ and $d = 8, 9, 10$ and 11. The orange and green horizontal lines are the values of $\log_b \lfloor c/b \rfloor$ and $\log_b \lceil c/b \rceil$, respectively.

Proposition 2.3. *Suppose that $b \mid d$, and denote $c = d/b$. Then*

$$\log a_n = \frac{\log c}{\log b} \log n + O(1).$$

Proof. Note that since $b \mid d$, the recursion formula for a_n simplifies slightly to

$$a_n = \sum_{\substack{0 \le r \le d-1 \\ r \equiv n \mod b}} a_{(n-r)/b} = \sum_{i=0}^{c-1} a_{\lfloor n/b \rfloor - i}. \tag{2.7}$$

From this we see that if $\lfloor \frac{m}{b} \rfloor = \lfloor \frac{n}{b} \rfloor$, then $a_n = a_m$; in fact, for $n \in \mathbb{N}$ we have that

$$a_n - a_{n-1} = \begin{cases} 0 & b \nmid n \\ a_{n/b} - a_{n/b - d/b} = a_{n/b} - a_{n/b-c} & b \mid n \end{cases}.$$

From this it is easy to show by induction on n that $a_n \ge a_{n-1}$ for all $n \in \mathbb{N}$.

Thus using the recursion formula Equation 2.7, we can establish the following inequalities:

$$c a_{\lfloor n/b \rfloor - c + 1} \le a_n \le c a_{\lfloor n/b \rfloor}.$$

Now the inequality $\log a_n \le \log c \log_b n$ is true for $n = 1$ since $a_1 = 1$, so suppose that $n > 1$ and that the inequality in question is true for $1 \le m < n$. Then

$$\begin{aligned} \log a_n &\le \log c + \log a_{\lfloor n/b \rfloor} \le (1 + \log_b \lfloor n/b \rfloor) \log c \\ &\le (1 + \log_b n/b) \log c = \log c \log_b n \end{aligned}$$

and so by induction $\log a_n \le \log c \log_b n$ for all $n \in \mathbb{N}$.

Let $M = \min \left\{ \log a_n - \log c \log_b \left(n + \frac{d}{b-1}\right) \mid n \in \{0, 1, \ldots, d-1\} \right\}$; this is defined as such so that the inequality $\log a_n \ge \log c \log_b(n + d/(b-1)) + M$ is true for $n \in \{1, 2, \ldots, d-1\}$. Then let $n \ge d$, and suppose that the inequality in question is true for $1 \le m < n$. Then $n > \lfloor n/b \rfloor - c + 1 \ge \lfloor d/b \rfloor - c + 1 = 1$, and so

$$\begin{aligned} \log a_n &\ge \log c + \log a_{\lfloor n/b \rfloor - c + 1} \\ &\ge \log c\,(1 + \log_b(\lfloor n/b \rfloor - c + 1 + d/(b-1))) + M \\ &\ge \log c\,(1 + \log_b(n/b - c + d/(b-1))) + M \\ &= \log c(\log_b(n + d/(b-1))) + M; \end{aligned}$$

hence by induction $\log a_n \ge \log c(\log_b(n + d/(b-1))) + M$ for all $n \in \mathbb{N}$.

The final ineffective asymptotic formula for a_n follows by consideration of the preceding inequalities. $\square$

Now we can finally prove the bounds mentioned earlier for p_d and q_d:

Corollary 2.3.1. *For $d \geq b$, $p_d \leq \log_b d - 1 \leq q_d$.*

Proof. Note that by the above proposition, $p_b = \log_b b - 1 = 0$ and $p_{bd} = \log_b(db) - 1 = \log_b d$. So we apply Theorem 2.2 with $d_1 = b$ and $d_2 = d$:

$$p_{bd} \geq p_b + p_d + 1 \iff p_d \leq p_{bd} - p_b - 1 = \log_b d - 0 - 1 = \log_b d - 1$$

as desired. The bound for q_d is proved similarly. □

Our next result extends the above proposition to the case where $b \nmid d$:

Theorem 2.4. *Suppose that $d \geq b$. Then*

$$n^{\log_b \lfloor d/b \rfloor} \ll a_n \ll n^{\log_b \lceil d/b \rceil}.$$

Proof. Recall that $a_{d,n}$ is nondecreasing in d. Hence by Proposition 2.3,

$$\begin{aligned} a_n = a_{d,n} \leq a_{b\lceil d/b \rceil,n} &\ll n^{\log_b \lceil d/b \rceil}, \quad \text{and similarly} \\ a_n \geq a_{b\lfloor d/b \rfloor,n} &\gg n^{\log_b \lfloor d/b \rfloor}. \end{aligned}$$

So the theorem is proved. □

Our first proof above shows that this theorem is in fact a simple corollary of Proposition 2.3, but we include an additional proof to illustrate a different method.

Proof. For brevity, define $f = \lfloor d/b \rfloor$ and $c = \lceil d/b \rceil > 1$. For each $n \in \mathbb{N}_0$, define $x_n = \frac{b(c-1)(b^n-1)}{b-1} \geq 0$, $y_n = 2cb^{n+1}$, $m_n = \min_{l \in [x_n, y_n)} a_l > 0$ and $M_n = \max_{l \in [x_n, y_n)} a_l > 0$. Note that $x_n \leq y_{n-1} < y_n$ for each n, that $x_n, y_n \to \infty$ as $n \to \infty$ and that with $n = \lceil \log_b(l/2bc) \rceil$, $x_n < l \leq y_n$ and so $m_n \leq a_l \leq M_n$. Moreover, for $l \in [x_n, y_n)$,

$$x_n \leq l \iff n \leq \log_b \left(1 + \frac{b-1}{b(c-1)} l\right) = \log_b l + O(1),$$

and similarly

$$l < y_n \iff n > \log_b l - \log_b(2bc)$$

and so $n = \log_b l + O(1)$.

Now let $n \in \mathbb{N}$ be large, and let $l \in [x_n, y_n)$ such that $a_l = m_n$. Since $a_{bk-1} \leq a_{bk-2} \leq \cdots \leq a_{bk-b}$, we can without loss of generality assume that

$l = bk - 1$ for some $k \in \mathbb{N}$. Note then that $k-1, k-2, \ldots, k-f \in [x_{n-1}, y_{n-1})$. Thus

$$m_n = a_l = a_{bk-1} = a_{k-1} + a_{k_2} + \cdots + a_{k-f} \geq f \times m_{n-1}$$

and so there is a $C > 0$ such that $m_n \geq Cf^n$ for sufficiently large n. Thus for sufficiently large $l \in \mathbb{N}$ and n such that $l \in [x_n, y_n)$,

$$a_l \geq m_n \geq Cf^n \gg f^{\log_b l} = l^{\log_b f}.$$

Similarly, let n be large and let $l \in [x_n, y_n)$ such that $a_l = M_n$. Without loss of generality, we can assume that $l = bk$ for some $k \in \mathbb{N}$. Note then that $k, k-1, \ldots, k-c+1 \in [x_{n-1}, y_{n-1})$. Thus

$$M_n = a_l = a_{bk} = a_k + a_{k-1} + \cdots + a_{k-c+1} \leq c \times M_{n-1}$$

and so there is a $D > 0$ such that $M_n \leq Dc^n$ for sufficiently large n. Thus for large l and n such that $l \in [x_n, y_n)$,

$$a_l \leq Dc^n \ll c^{\log_b l} = l^{\log_b c}. \qquad \square$$

Now henceforth we denote by $\lceil x \rceil$ the least integer not less than x as per the usual definition, *except* in the case that $x = \frac{1}{b}$; then we define $\left\lceil \frac{1}{b} \right\rceil = \frac{1}{b}$. Then by the above theorem and the fact that a_m is bounded for $d \leq b$ and eventually zero for $d = 1$, we have that $a_m \ll m^{\log_b \lceil d/b \rceil}$ for all $d \in \mathbb{N}$. Also note that then for each $d \in \mathbb{N}$, $d/b \leq \lceil d/b \rceil < d$.

Now we have shown that when $b \mid d$, $p_d = q_d = \log_b d - 1$, and that in general $\log_b \lfloor d/b \rfloor \leq p_d \leq q_d \leq \log_b \lceil d/b \rceil$. However, the question of when p_d and q_d are equal has not yet been fully answered. The following theorem finished the job, showing that this *only* happens when $b \mid d$:

Theorem 2.5. *$p_d = q_d \implies b \mid d$.*

The following proof is due to Prof. Stephan Wagner:

Proof. For $m \in \mathbb{N}$, we define the column vector $v_m = (a_{m-1}, a_{m-2}, \ldots, a_{m-p})^\top$ where $p = \left\lceil \frac{d-b}{b-1} \right\rceil$.[§] Then the recurrence equation

$$a_m = \sum_{\substack{0 \leq r < d \\ r \equiv m \mod b}} a_{\frac{m-r}{b}}$$

[§]For any matrix M, we denote its transpose by $M^\top$.

implies that $v_{bm} = Tv_m$, where T is the transfer matrix with entries given by $T_{ij} = \begin{cases} 1 & \text{if } bj - i \in \{0, 1, \ldots, d-1\} \\ 0 & \text{otherwise} \end{cases}$ for $1 \leq i, j \leq p$.

Thus for $k \in \mathbb{N}$, $v_{b^k} = T^k v_1$ and so

$$a_{b^k - 1} = \begin{pmatrix} 1 & 0 \ldots & 0 \end{pmatrix} v_{b^k} = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} T^k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.8}$$

Now for $1 \leq i, j \leq p$ with $|i - j| \leq 1$,

$$\begin{aligned} bj - i = (b-1)j + (j-1) &\in [(b-1)j - 1, (b-1)j + 1] \\ &\subseteq [(b-1) \times 1 - 1, (b-1)p + 1] \\ &\subseteq [b - 2, d - b + b - 1 + 1) \\ &\subseteq [0, d) \end{aligned}$$

and so $T_{ij} = 1$. Hence T^k has only positive entries for some $k \in \mathbb{N}$¶, and so T is *primitive*. Thus by the Perron-Frobenius Theorem‖, $T^k \sim \rho(T)^k q r^\top$ for large k where $\rho(T)$ is the spectral radius of T (which in this case is its largest eigenvalue) and $q, r > 0$ are Perron vectors for T and $T^\top$ respectively. Applying this to Equation 2.8 yields

$$\begin{aligned} a_{b^k - 1} &\sim \rho(T)^k \begin{pmatrix} 1 & \ldots & 0 \end{pmatrix} q r^\top \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = \rho(T)^k q_1 r_1, \\ \implies \frac{\log a_{b^k - 1}}{\log(b^k - 1)} &= \frac{k \log \rho(T) + O(1)}{k \log b + o(1)} = \log_b \rho(T) + O\left(k^{-1}\right). \end{aligned}$$

Now if $p_d = q_d$, then by Corollary 2.3.1,

$$\log_b \frac{d}{b} = p_d = q_d = \lim_{m \to \infty} \frac{\log a_m}{\log m} = \lim_{k \to \infty} \frac{\log a_{b^k - 1}}{\log(b^k - 1)} = \log_b \rho(T)$$

$$\implies \rho(T) = \frac{d}{b}.$$

But T has integer coefficients and so its characteristic polynomial is monic with integer coefficients, so $\rho(T)$ (being a root of the characteristic polynomial of T) is an algebraic integer. So $\frac{d}{b}$, being rational, is an algebraic integer and thus an integer. Thus $b \mid d$, as desired. □

¶In particular, k may be taken to be $p - 1$.

‖see, for instance Meyer [16, p. 674]

2.3 Limiting behaviour of $\frac{a_m}{m^{\log_b c}}$

So we see from the above that when $b \mid d$, with $c := d/b$, $a_n = \Theta\left(n^{\log_b c}\right)$. So the question of the behaviour of the quotient $a_n/n^{\log_b c}$ arises. Figure 2.3 and Figure 2.4 show plots of this quantity for $b = 4$ and c between 2 and 9 (when $c = 1$, $a_n = 1$ for all $n \in \mathbb{N}$ and so the corresponding plot would be rather uninteresting):

These plots suggest that $a_m/m^{\log_b c}$ is asymptotically a periodic function of $\ln m$, more specifically a periodic function of $\log_b m$ of period 1**. Before we prove this, we will need the following auxiliary lemma, the proof of which is delegated to Appendix B:

Lemma 2.1. *Let $b \in \mathbb{N}$ be greater than 1, let $\eta > 0$ and let $(x_m)_{m\in\mathbb{N}}$ be a sequence of real numbers such that $x_m - x_{bm} \ll m^{-\eta}$ and $x_{m+1} - x_m \ll m^{-\eta}$ for large m. Then there is a 1-periodic continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $x_m = f(\log_b m) + O(m^{-\eta})$ for large m. Moreover, if $\eta \leq 1$, then the function f is Hölder continuous with exponent η, and if $\eta > 1$ then f is constant. In particular, if $\eta = 1$ then f is Lipschitz continuous.*

Now we apply the above lemma to our sequence a_m, under the assumption that d is a multiple of b:

Theorem 2.6. *Suppose that $b \mid d$, and let $c = d/b$. Then there is a Hölder continuous 1-periodic function f_c with exponent $\log_b(c/\lceil c/b\rceil)$ such that*

$$a_m = m^{\log_b c} f_c(\log_b m) + O\left(m^{\log_b \lceil c/b\rceil}\right).$$

In particular, if $b \mid c$ then the function f_c is Lipschitz continuous.

Proof. If $d = b$, then $a_m = 1$ identically and so the theorem holds. So henceforth we assume that $d > b$, i.e. $c \geq 2$. By Theorem 2.4 we have that $a_m = \Theta\left(m^{\log_b c}\right)$; also, since $a_m - a_{m-1} = a_{c,m/b}$ if $b \mid m$ and 0 otherwise, $a_m - a_{m-1} \ll m^{\log_b \lceil c/b\rceil}$. Thus, with $\eta = \log_b(c/\lceil c/b\rceil)$ and $x_m = a_m/m^{\log_b c}$,

$$\begin{aligned} &ca_m - a_{bm} = a_m - a_{m-1} + a_m - a_{m-2} + \cdots + a_m - a_{m-c+1} \ll m^{\log_b \lceil c/b\rceil} \\ \iff\ &x_m - x_{bm} = \frac{a_m}{m^{\log_b c}} - \frac{a_{bm}}{(bm)^{\log_b c}} \ll m^{\log_b(\lceil c/b\rceil/c)} = m^{-\eta}. \end{aligned}$$

**Henceforth, a periodic function of period 1 will be called 1-periodic.

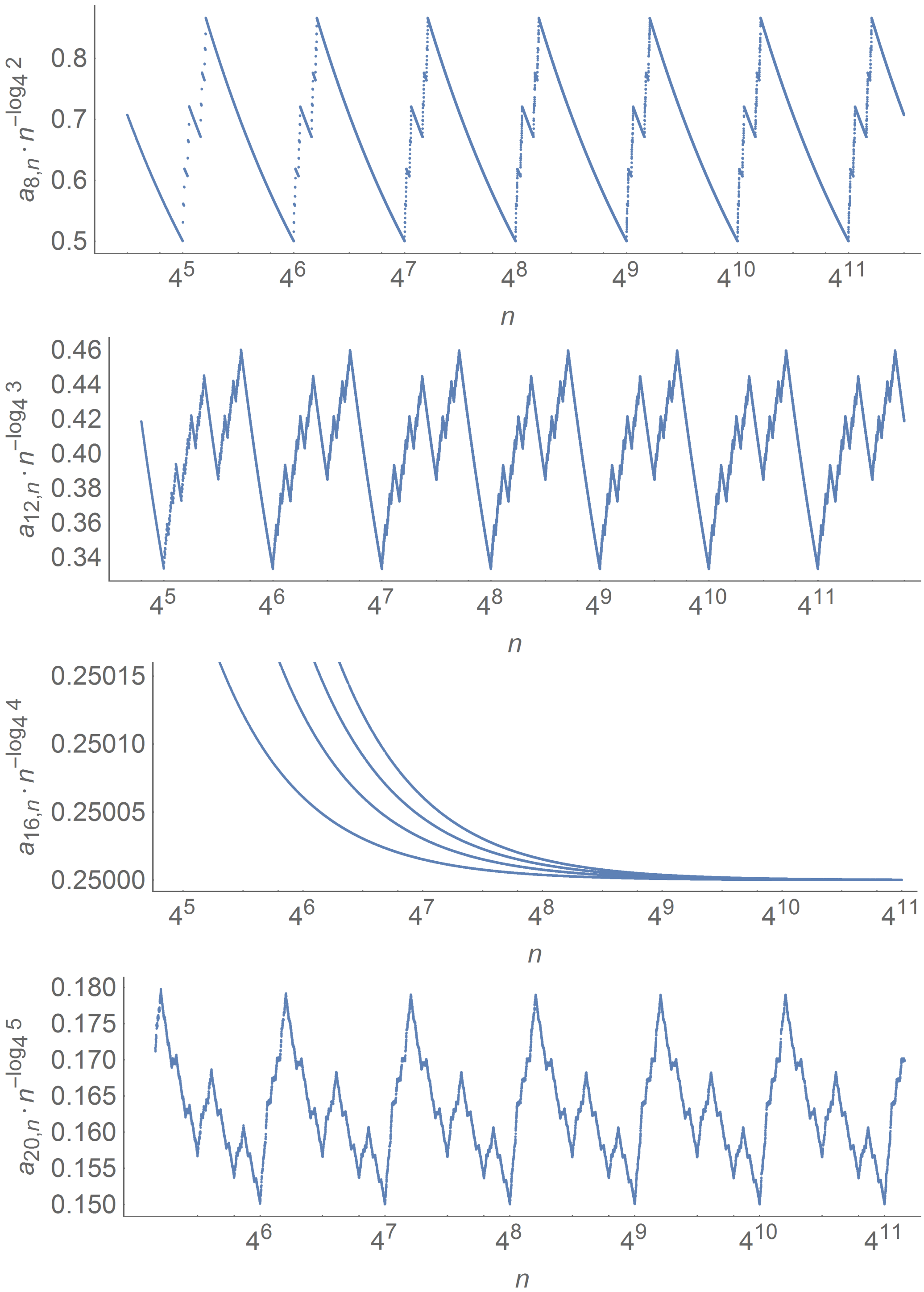

Figure 2.3: Plots of $a_{bc,n}/n^{\log_b c}$ versus n (n being on a logarithmic scale) for $b = 4$ and $c = 2, 3, 4$ and 5.

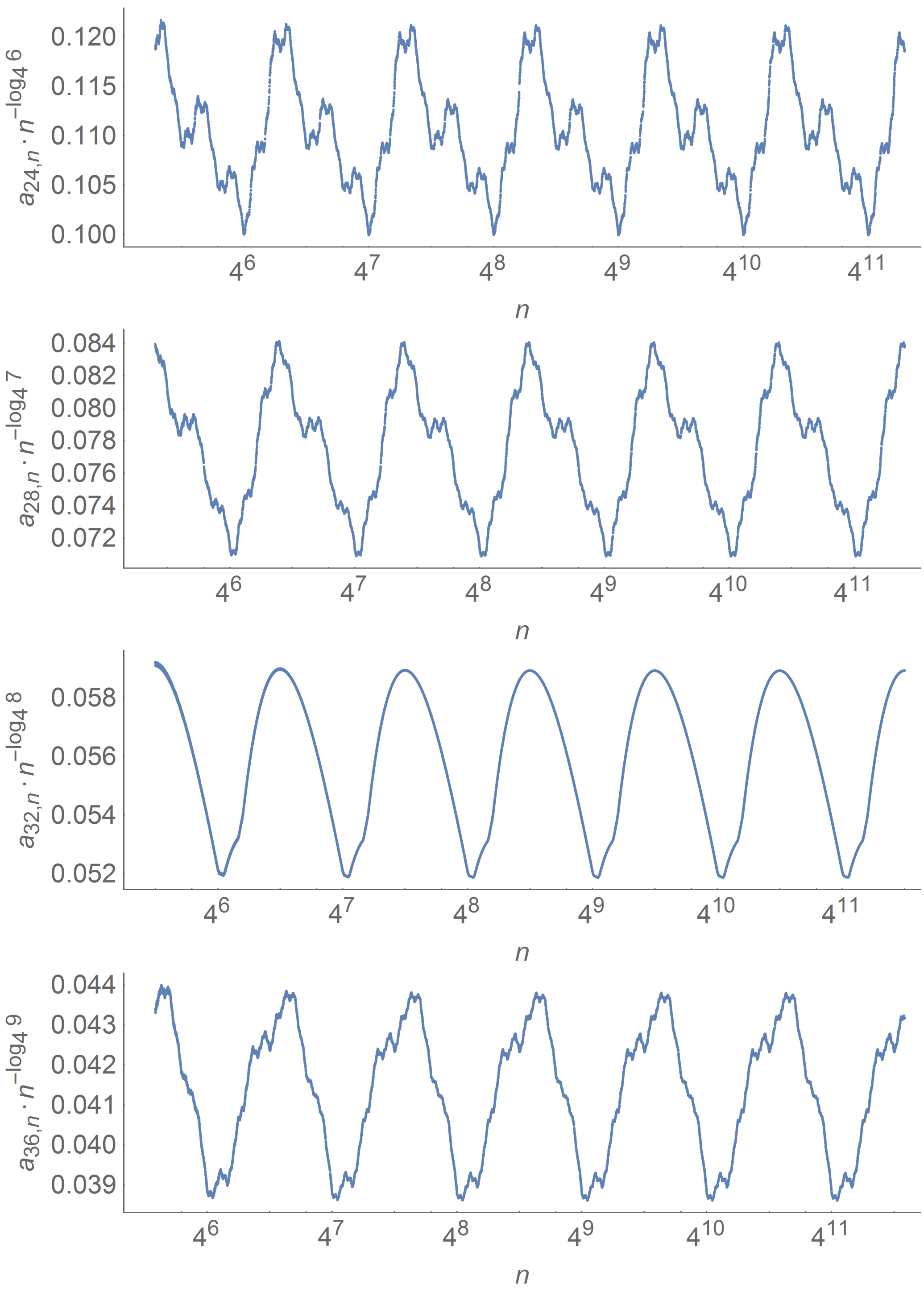

Figure 2.4: Plots of $a_{bc,n}/n^{\log_b c}$ versus n (n being on a logarithmic scale) for $b = 4$ and $c = 6, 7, 8$ and 9.

Similarly,

$$\begin{aligned}
x_{m+1} - x_m &= \frac{a_{m+1}}{(m+1)^{\log_b c}} - \frac{a_m}{m^{\log_b c}} \\
&= \frac{a_{m+1} - a_m}{(m+1)^{\log_b c}} + a_m \left(\frac{1}{(m+1)^{\log_b c}} - \frac{1}{m^{\log_b c}} \right) \\
&\ll m^{\log_b \lceil c/b \rceil / c} + m^{-1} \\
&\ll m^{-\eta} \quad \text{since } \eta = \log_b \frac{c}{\lceil c/b \rceil} \le \log_b \frac{c}{c/b} = 1.
\end{aligned}$$

So the sequence x_m satisfies the conditions of Lemma 2.1 with the constant

$$\eta = \log_b \frac{c}{\lceil c/b \rceil} \ge \log_b \frac{c}{\lceil c/2 \rceil} > \log_b \frac{c}{\frac{c}{2}+1} = \log_b \frac{2c}{c+2} \ge \log_b \frac{2c}{c+c} = 0.$$

Hence there is a Hölder continuous 1-periodic function f_c with exponent $\eta = \log_b(c/\lceil c/b \rceil)$ such that

$$\begin{aligned}
& x_m = f_c(\log_b m) + O\left(m^{\log_b(\lceil c/b \rceil / c)}\right) \\
\iff\ & a_m = m^{\log_b c} f_c(\log_b m) + O\left(m^{\log_b \lceil c/b \rceil}\right),
\end{aligned}$$

as desired. □

We henceforth denote the periodic function appearing in the above theorem by f_c for each $c \in \mathbb{N}$.

2.4 Smoothness of the limiting periodic function f_c

In the preceding theorem, we showed that when $c = d/b \in \mathbb{N}$, a_m is asymptotically the product of a power of m (specifically $m^{\log_b c}$) and a Hölder continuous 1-periodic function f_c. Now we focus on the periodic function f_c, particularly on its smoothness properties and relationships between the f_c for different c.

In the case that $b \mid c$, we have from Theorem 2.6 that f_c is Lipschitz continuous. However, much more can be said. In this case, f_c is actually continuously differentiable. Not only that, but there is a differential relationship between f_c and $f_{c/b}$, as the next theorem shows.

Theorem 2.7. *Suppose that $b \mid c$. Then f_c is continuously differentiable, and moreover*

$$\frac{\mathrm{d}}{\mathrm{d}x} f_c(x) = -\log c \cdot f_c(x) + \frac{\log b}{c} f_{c/b}(x). \tag{2.9}$$

Proof. Let $c = b\tilde{c}$ where $\tilde{c} \in \mathbb{N}$. We continue with the notation used in the proofs of Lemma 2.1 and Theorem 2.6.

There we defined an auxiliary function g by linear interpolation between the points $(\log_b m, x_m)$ where $x_m = a_m m^{-\log_b c}$, which satisfies the property $g(x) - f_c(x) \ll b^{-x}$. However, since $x_m - x_{m-1} \ll m^{-1}$[††], the reader should have no trouble seeing that the alternative function $\tilde{g}$ defined via linear interpolation between the points $(\log_b m, x_{bm})$ also satisfies the same property. Also, we (re)denote $\eta = \log_b(\tilde{c}/\lceil \tilde{c}/b \rceil)$, noting that $\eta \in (0,1]$.

First, note that

$$\begin{aligned} a_{bm} - a_{b(m-1)} &= a_m + a_{m-1} + \cdots + a_{m-c+1} - a_{m-1} - a_{m-2} - \cdots - a_{m-c} \\ &= a_m - a_{m-c} = a_{c,m}, \end{aligned}$$

the last equality coming from examination of the associated generating functions. Now let $m \in \mathbb{N}$ be large. Then

$$\begin{aligned}
&\tilde{g}(\log_b m) - \tilde{g}(\log_b(m-1)) \\
&= x_{bm} - x_{b(m-1)} = \frac{a_{bm}}{(bm)^{\log_b c}} - \frac{a_{b(m-1)}}{(bm-b)^{\log_b c}} \\
&= a_{b(m-1)} \left(\frac{1}{cm^{\log_b c}} - \frac{1}{c(m-1)^{\log_b c}} \right) + \frac{a_{bm} - a_{b(m-1)}}{cm^{\log_b c}} \\
&= \Big((b(m-1))^{\log_b c} f_c(\log_b(bm-b)) + O\big(m^{\log_b c - 1}\big) \Big) \\
&\qquad \times \left(\frac{-\log_b c}{cm^{\log_b c + 1}} + O\big(m^{-\log_b c - 2}\big) \right) + \frac{a_{c,m}}{cm^{\log_b c}} \\
&= \left(f_c(\log_b(m-1)) + O\Big(\frac{1}{m}\Big) \right) \left(\frac{-\log_b c\ (m-1)^{\log_b c}}{m^{\log_b c + 1}} + O\Big(\frac{1}{m^2}\Big) \right) \\
&\qquad + \frac{1}{cm^{\log_b c}} \Big(m^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big) \Big) \\
&= \frac{-\log_b c}{m} f_c(\log_b(m-1)) + O\Big(\frac{1}{m^2}\Big) + \frac{f_{\tilde{c}}(\log_b m)}{cm} + O\big(m^{\log_b(\lceil \tilde{c}/b \rceil / c)}\big) \\
&= -\log c\ f_c(\log_b(m-1))\,(\log_b m - \log_b(m-1)) + O\Big(\frac{1}{m^2}\Big) \qquad (2.11) \\
&\qquad + \frac{\log b}{c} f_{\tilde{c}}(\log_b m)\,(\log_b m - \log_b(m-1)) + O\big(m^{-\eta-1}\big)
\end{aligned}$$

[††]since here the relevant $\eta = \log_b \frac{c}{c/b} = 1$

$$= O\left(\frac{1}{m}\right);$$

since $\eta \leq 1$ the error terms $O(m^{-2})$ and $O(m^{-\eta-1})$ in the second-to-last expression can be combined into one $O(m^{-\eta-1})$ term.

Now let x and y be real numbers with $x < y$, and let $n \in \mathbb{N}$ be large. Then there are unique large $k, l \in \mathbb{N}$ such that $n + x \in (\log_b(k-1), \log_b k]$ and $n + y \in (\log_b(l-1), \log_b l]$. Note that $k = b^{n+x} + O(1)$ and $l = b^{n+y} + O(1)$; thus using Equation 2.11 we have

$$\begin{aligned}
&\tilde{g}(n+y) - \tilde{g}(n+x) \\
&= \tilde{g}(n+y) - \tilde{g}(\log_b(l-1)) + \tilde{g}(\log_b k) - \tilde{g}(n+x) \\
&\qquad + \sum_{m=k+1}^{l-1} \tilde{g}(\log_b m) - \tilde{g}(\log_b(m-1)) \\
&= \frac{n+y-\log_b(l-1)}{\log_b l - \log_b(l-1)} \left(\tilde{g}(\log_b l) - \tilde{g}(\log_b(l-1))\right) \\
&\quad + \frac{\log_b k - n - x}{\log_b k - \log_b(k-1)} \left(\tilde{g}(\log_b k) - \tilde{g}(\log_b(k-1))\right) \\
&\qquad + \sum_{m=k+1}^{l} \tilde{g}(\log_b m) - \tilde{g}(\log_b(m-1)) \\
&= O\left(\frac{1}{l}\right) + O\left(\frac{1}{k}\right) - \log c \sum_{m=k+1}^{l-1} f_c(\log_b(m-1)) \left(\log_b m - \log_b(m-1)\right) \\
&\quad + \frac{\log b}{c} \sum_{m=k+1}^{l-1} f_{\tilde{c}}(\log_b m) \left(\log_b m - \log_b(m-1)\right) + \sum_{m=k+1}^{l-1} O\left(m^{-\eta-1}\right) \\
&= -\log c \sum_{m=k+1}^{l-1} f_c(\log_b(m-1) - n) \left(\log_b m - \log_b(m-1)\right) \qquad (2.12) \\
&\quad + \frac{\log b}{c} \sum_{m=k+1}^{l-1} f_{\tilde{c}}(\log_b m - n) \left(\log_b m - \log_b(m-1)\right) \\
&\qquad + O\left(b^{-n-y}\right) + O\left(b^{-n-x}\right) + O\left(k^{-\eta} - l^{-\eta}\right),
\end{aligned}$$

the last error term above being $O(b^{-n\eta})$ since $\eta \leq 1$.

Now as $n \to \infty$ these three error terms vanish. On the other hand, since f_c and $f_{\tilde{c}}$ are continuous they are Riemann integrable, and so since $\log_b k - n \to x$

and $\log_b(l-1) - n \to y$ as $n \to \infty$ we have that

$$\sum_{m=k+1}^{l-1} f_c(\log_b(m-1) - n)\,(\log_b m - \log_b(m-1)) \to \int_x^y f_c(t)\mathrm{d}t \quad \text{as } n \to \infty \tag{2.13}$$

and

$$\sum_{m=k+1}^{l-1} f_{\tilde{c}}(\log_b m - n)\,(\log_b m - \log_b(m-1)) \to \int_x^y f_{\tilde{c}}(t)\mathrm{d}t \quad \text{as } n \to \infty. \tag{2.14}$$

Finally, $\tilde{g}(n+y) - \tilde{g}(n+x) \to f_c(y) - f_c(x)$ as $n \to \infty$, and so putting this together with Equation 2.12, Equation 2.13 and Equation 2.14 we get

$$f_c(y) - f_c(x) = -\log c \int_x^y f_c(t)\mathrm{d}t + \frac{\log b}{c} \int_x^y f_{\tilde{c}}(t)\mathrm{d}t.$$

Considering y to be variable , we see from the above that $f_c(y)$ is an antiderivative of a continuous function, and is hence continuously differentiable by the Fundamental Theorem of Calculus. Dividing by $y - x$ and sending y to x, we then arrive at the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} f_c(x) = -\log c \cdot f_c(x) + \frac{\log b}{c} f_{\tilde{c}}(x)$$

as desired. □

Corollary 2.7.1. *Suppose that $b^n \mid c$ for $n \in \mathbb{N}_0$. Then f_c is n times continuously differentiable, and $f_c^{(n)}$ is a linear combination of $f_c, f_{c/b}, \ldots, f_{c/b^n}$, and is thus Hölder continuous of exponent $\log_b \frac{c/b^n}{\lceil c/b^{n+1} \rceil}$.*

Proof. We proceed by induction on n. The case $n = 0$ follows from Theorem 2.6, and the case $n = 1$ follows from Theorem 2.7.

Now suppose that the above corollary has been proven for some $n \in \mathbb{N}$, and let $c \in \mathbb{N}$ be such that $b^{n+1} \mid c$. Then by the induction hypothesis, f_c and $f_{c/b}$ are n times continuously differentiable. So we differentiate Equation 2.9 $n-1$ times with respect to x to obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f_c(x) = -\log c \, \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} f_c(x) + \frac{\log_b}{c} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} f_{c/b}(x).$$

Now the right-hand side of the above equation is differentiable by the induction hypothesis, and so f_c is $n+1$ times differentiable. Differentiating the above equation, we then have

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} f_c(x) = -\log c \, \frac{\mathrm{d}^n}{\mathrm{d}x^n} f_c(x) + \frac{\log_b}{c} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f_{c/b}(x).$$

Now by the induction hypothesis, $f_c^{(n)}$ is a linear combination of $f_c, \dots, f_{c/b^n}$ and $f_{c/b}^{(n)}$ is linear combination of $f_{c/b}, f_{c/b^2}, \dots, f_{c/b^{n+1}}$. Hence $f_c^{(n+1)}$ is a linear combination of $f_c, f_{c/b}, \dots, f_{c/b^{n+1}}$ and thus Hölder continuous of exponent $\log_b \frac{c/b^{n+1}}{\lceil c/b^{n+2} \rceil}$ by Theorem 2.6, as desired. So the induction step is proved, and the result holds for all $n \in \mathbb{N}_0$. $\square$

Corollary 2.7.2. *If $b \mid c$ then, for $x, y \in \mathbb{R}$ with $y - x$ small,*

$$f_c(y) - f_c(x) = \left(-\log c \; f_c(x) + \frac{\log b}{c} f_{c/b}(x) \right) (y - x) + O\left(|y - x|^{\log_b \left(c / \lceil c/b^2 \rceil \right)} \right).$$

Proof. We simply apply Theorem 2.7 and Lemma A.1 with $n = 1$, $f = f_c$ and $r = \log_b \left(c \,/ b \lceil c/b^2 \rceil \right)$. $\square$

2.5 More precise asymptotic expansions

We can obtain a better approximation to a_n than Theorem 2.6 in the case that $b \mid c$, but before proving this result, we will need the following auxiliary result, the proof of which we delegate to Appendix B:

Lemma 2.2. *Let $b \in \mathbb{N} \backslash \{1\}$, let $\eta \in (0, 1]$ and let $(x_m)_{m \in \mathbb{N}}$ be a sequence of real numbers such that*

(a) $x_m - x_{bm} \ll m^{-\eta}$

(b) $x_m - x_{m-1} \ll m^{-\eta}$

(c) $x_m - (b + 1) x_{bm} + b x_{b^2 m} \ll m^{-1-\eta}$

(d) $x_m - x_{bm} - x_{m-1} + x_{bm-b} \ll m^{-1-\eta}$

Then there are 1-periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $x_m = f(\log_b m) + \frac{1}{m} g(\log_b m) + O(m^{-1-\eta})$ for large m.

Now we apply the above lemma:

Theorem 2.8. *If $b \mid c$, then for large m,*

$$a_{bm-1} = cm^{\log_b c} f_c(\log_b m) + \frac{c-b}{2b-2} m^{\log_b c-1} f_{c/b}(\log_b m) + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big)$$

Proof. Our proof is similar to that of Theorem 2.6, but instead of using Lemma 2.1 we use Lemma 2.2.

We denote c/b by $\tilde{c}$, and $x_m = a_{bm-1}/m^{\log_b c}$. Now

$$\begin{aligned}
&x_m - x_{bm}\\
&= \frac{a_{bm-1}}{m^{\log_b c}} - \frac{a_{b^2m-1}}{(bm)^{\log_b c}} = \frac{a_{bm-1}}{m^{\log_b c}} - \frac{a_{b(bm-1)}}{cm^{\log_b c}}{}^{\ddagger\ddagger}\\
&= \frac{1}{cm^{\log_b c}} \left(ca_{bm-1} - a_{bm-1} - a_{bm-2} - \cdots - a_{bm-c}\right)\\
&= \frac{1}{cm^{\log_b c}} \left(ca_{bm-b} - ba_{bm-b} - ba_{bm-2b} - \cdots - ba_{bm-b\tilde{c}}\right)\\
&= \frac{1}{\tilde{c}m^{\log_b c}} \big((\tilde{c}-1)(a_{bm-b} - a_{bm-2b}) + (\tilde{c}-2)(a_{bm-2b} - a_{bm-3b}) +\\
&\qquad + \cdots + 2(a_{bm-(\tilde{c}-2)b} - a_{bm-(\tilde{c}-1)b}) + (a_{bm-(\tilde{c}-1)b} - a_{bm-\tilde{c}b})\big)\\
&= \frac{1}{\tilde{c}m^{\log_b c}} \left((\tilde{c}-1)a_{c,m-1} + (\tilde{c}-2)a_{c,m-2} + \cdots + a_{c,m-\tilde{c}+1}\right)\\
&= \frac{1}{\tilde{c}m^{\log_b c}} \Big((\tilde{c}-1)\left((m-1)^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b(m-1)) + O\big((m-1)^{\log_b \lceil \tilde{c}/b \rceil}\big)\right)\\
&\qquad + (\tilde{c}-2)\left((m-2)^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b(m-2)) + O\big((m-2)^{\log_b \lceil \tilde{c}/b \rceil}\big)\right)\\
&\qquad + \cdots\\
&\qquad + (m-\tilde{c}+1)^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b(m-\tilde{c}+1)) + O\big((m-\tilde{c}+1)^{\log_b \lceil \tilde{c}/b \rceil}\big)\Big).
\end{aligned}$$

Now for each $k \in \{1, 2, \ldots, \tilde{c}-1\}$, since $f_{\tilde{c}}$ is Hölder continuous with exponent $\log_b(\tilde{c}/\lceil \tilde{c}/b \rceil)$,

$$\begin{aligned}
&(m-k)^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b(m-k))\\
&= \left(m^{\log_b \tilde{c}} + O\big(m^{\log_b \tilde{c}-1}\big)\right)\left(f_{\tilde{c}}(\log_b m) + O\big((1/m)^{\log_b(\tilde{c}/\lceil \tilde{c}/b \rceil)}\big)\right)\\
&= m^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \tilde{c}-1}\big) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil - 1}\big)\\
&= m^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big).
\end{aligned}$$

‡‡see Equation 2.5

Hence

$$\begin{aligned}
&x_m - x_{bm}\\
&= \frac{1}{\tilde{c}m^{\log_b c}} \left(\left(m^{\log_b \tilde{c}} f_{\tilde{c}}(\log_b m) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil}\right)\right) \left((\tilde{c}-1) + (\tilde{c}-2) + \cdots + 1\right)\right)\\
&= \frac{\tilde{c}-1}{2m} f_{\tilde{c}}(\log_b m) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right), \qquad (2.15)
\end{aligned}$$

so

$$\begin{aligned}
&x_m - (b+1)x_{bm} + bx_{b^2m}\\
&= x_m - x_{bm} - b(x_{bm} - x_{b^2m})\\
&= \frac{\tilde{c}-1}{2m} f_{\tilde{c}}(\log_b m) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right) - \frac{b\tilde{c}-b}{2bm} f_{\tilde{c}}(\log_b bm) + O\left((bm)^{\log_b \lceil \tilde{c}/b \rceil / c}\right)\\
&= \frac{\tilde{c}-1}{2m} \left(f_{\tilde{c}}(\log_b m) - f_{\tilde{c}}(\log_b m)\right) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right)\\
&= O\left(m^{-1+\log_b \lceil \tilde{c}/b \rceil / \tilde{c}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&x_m - x_{bm} - x_{m-1} + x_{bm-b}\\
&= \frac{\tilde{c}-1}{2m} f_{\tilde{c}}(\log_b m) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right) - \frac{\tilde{c}-1}{2(m-1)} f_{\tilde{c}}(\log_b (m-1))\\
&\qquad + O\left((m-1)^{\log_b \lceil \tilde{c}/b \rceil / c}\right)\\
&= \left(\frac{\tilde{c}-1}{2m} - \frac{\tilde{c}-1}{2m-2}\right) f_{\tilde{c}}(\log_b m) + \frac{\tilde{c}-1}{2m-2} \left(f_{\tilde{c}}(\log_b m) - f_{\tilde{c}}(\log_b (m-1))\right)\\
&\qquad + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right)\\
&= O\left(m^{-2}\right) f_{\tilde{c}}(\log_b m) + \frac{\tilde{c}-1}{2m-2} O\left((1/m)^{\log_b (\tilde{c}/\lceil \tilde{c}/b \rceil)}\right) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil / c}\right)\\
&= O\left(m^{-1+\log_b \lceil \tilde{c}/b \rceil / \tilde{c}}\right).
\end{aligned}$$

Finally, as in the proof of Theorem 2.6 we have that $x_m - x_{m-1} \ll m^{-1}$ and $x_m - x_{bm} \ll m^{-1}$. Hence x_m satisfies the conditions of Lemma 2.2 with $\eta = \lceil \tilde{c}/b \rceil / \tilde{c}$, and so there are 1-periodic continuous functions $\hat{f}$ and $\hat{g}$ such that

$$\begin{aligned}
&x_m = \hat{f}(\log_b m) + \frac{1}{m} \hat{g}(\log_b m) + O\left(m^{-1+\log_b \lceil \tilde{c}/b \rceil / \tilde{c}}\right)\\
\iff\ & a_{bm-1} = m^{\log_b c} \hat{f}(\log_b m) + m^{\log_b c - 1} \hat{g}(\log_b m) + O\left(m^{\log_b \lceil \tilde{c}/b \rceil}\right).
\end{aligned} \qquad (2.16)$$

Now by Theorem 2.6, we have that

$$\begin{aligned} a_{bm-1} &= (bm-1)^{\log_b c} f_c(\log_b(bm-1)) + O\big((bm-1)^{\log_b c-1}\big) \\ &= (bm)^{\log_b c} f_c(\log_b bm) + O\big(m^{\log_b c-1}\big) \text{ since } f_c \text{ is Lipschitz continuous} \\ &= cm^{\log_b c} f_c(\log_b m) + O\big(m^{\log_b c-1}\big). \end{aligned}$$

Comparing this to Equation 2.16, we see that $\hat{f} = cf_c$.

Moreover, by Equation 2.15 we have that

$$\begin{aligned} x_m - x_{bm} &= \frac{\tilde{c}-1}{2m} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil / c}\big) \\ \iff ca_{bm-1} - a_{b^2m-1} &= \frac{c(\tilde{c}-1)}{2} m^{\log_b c-1} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big), \end{aligned}$$

whereas Equation 2.16 gives

$$\begin{aligned} &ca_{bm-1} - a_{b^2m-1} \\ &= cm^{\log_b c} \hat{f}(\log_b m) + cm^{\log_b c-1} \hat{g}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big) \\ &\quad - (bm)^{\log_b c} \hat{f}(\log_b bm) - (bm)^{\log_b c-1} \hat{g}(\log_b bm) + O\big((bm)^{\log_b \lceil \tilde{c}/b \rceil}\big) \\ &= \left(c - \frac{c}{b}\right) m^{\log_b c-1} \hat{g}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big). \end{aligned}$$

Comparing these, we see that

$$\begin{aligned} \frac{c(b-1)}{b} \hat{g} &= \frac{c(\tilde{c}-1)}{2} f_{\tilde{c}} \\ \iff \hat{g} &= \frac{c-b}{2b-2} f_{\tilde{c}}, \end{aligned}$$

and so finally

$$a_{bm-1} = cm^{\log_b c} f_c(\log_b m) + \frac{c-b}{2b-2} m^{\log_b c-1} f_{\tilde{c}}(\log_b m) + O\big(m^{\log_b \lceil \tilde{c}/b \rceil}\big)$$

as desired. □

Note that we cannot have a general asymptotic expansion for a_m of the form

$$a_m = m^{\log_b c} f(\log_b m) + m^{\log_b c-1} g(\log_b m) + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big) \tag{2.17}$$

where g is Hölder continuous and f is differentiable with Hölder continuous derivative, since we would then have an asymptotic expansion for $a_m - a_{m-1}$ of the form

$$a_m - a_{m-1} = m^{\log_b c-1} h(\log_b m) + O\big(m^{\log_b \mu}\big) \quad \text{where } \mu < \log_b c - 1$$

and $h(\log_b m)$ is Hölder continuous, but $a_m = a_{m-1} = \cdots = a_{b\lfloor m/b\rfloor}$ and $a_{bm} - a_{bm-1} = a_{bm} - a_{bm-b} = a_{c,m} = \Theta\left(m^{\log_b c-1}\right)$. However, we can use the above result to obtain an asymptotic expansion similar to the above.

To this end, for m large we denote $\epsilon = \left\lfloor \frac{m}{b} \right\rfloor + 1 - \frac{m}{b} = 1 - \left\{\frac{m}{b}\right\} \in (0, 1]$. Here, and elsewhere, for $x \in \mathbb{R}$ we denote by $\{x\}$ the fractional part of x: $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer not greater than x. Then

$$\begin{aligned}
&a_m = a_{b\lfloor m/b\rfloor} = a_{b(\lfloor m/b\rfloor+1)-1} \\
&= c\left(\left\lfloor\frac{m}{b}\right\rfloor+1\right)^{\log_b c} f_c\left(\log_b\left(\left\lfloor\frac{m}{b}\right\rfloor+1\right)\right) + \frac{c-b}{2b-2}\left(\left\lfloor\frac{m}{b}\right\rfloor+1\right)^{\log_b c-1} \\
&\qquad\qquad \times f_{c/b}\left(\log_b\left(\left\lfloor\frac{m}{b}\right\rfloor+1\right)\right) + O\left(\left(\left\lfloor\frac{m}{b}\right\rfloor+1\right)^{\log_b\lceil c/b^2\rceil}\right) \\
&= c\left(\frac{m}{b}+\epsilon\right)^{\log_b c} f_c\left(\log_b\left(\frac{m}{b}+\epsilon\right)\right) \\
&\quad + \frac{c-b}{2b-2}\left(\frac{m}{b}+\epsilon\right)^{\log_b c-1} f_{c/b}\left(\log_b\left(\frac{m}{b}+\epsilon\right)\right) + O\left(m^{\log_b\lceil c/b^2\rceil}\right) \\
&= c\left(\left(\frac{m}{b}\right)^{\log_b c} + \epsilon\log_b c\left(\frac{m}{b}\right)^{\log_b c-1} + O\left(m^{\log_b c-2}\right)\right) f_c\left(\log_b(m+b\epsilon)\right). \\
&\qquad + \frac{c-b}{2b-2}\left(\left(\frac{m}{b}\right)^{\log_b c-1} + O\left(\left(\frac{m}{b}\right)^{\log_b c-2}\right)\right) f_{c/b}\left(\log_b(m+b\epsilon)\right) \\
&\qquad\qquad + O\left(m^{\log_b\lceil c/b^2\rceil}\right)
\end{aligned}$$

Now $f_{c/b}$ is Hölder continuous with exponent $\log_b\left(c/b\left\lceil c/b^2\right\rceil\right)$, and so

$$\begin{aligned}
f_{c/b}(\log_b(m+b\epsilon)) &= f_{c/b}(\log_b m) + O\left((1/m)^{\log_b\left(c/b\left\lceil c/b^2\right\rceil\right)}\right) \\
&= f_{c/b}(\log_b m) + O\left(m^{1+\log_b\lceil c/b^2\rceil/c}\right).
\end{aligned}$$

Similarly, by Theorem 2.7 and Corollary 2.7.2,

$$\begin{aligned}
f_c(\log_b(m+b\epsilon)) &= f_c(\log_b m) + \log_b\left(1+\frac{b\epsilon}{m}\right) f_c'(\log_b m) + O\left(m^{\log_b\lceil c/b^2\rceil/c}\right) \\
&= f_c(\log_b m) + \frac{b\epsilon}{m\log b} f_c'(\log_b m) + O\left(m^{\log_b\lceil c/b^2\rceil/c}\right).
\end{aligned}$$

Thus

$$
\begin{aligned}
a_m &= c\left(\frac{m}{b}\right)^{\log_b c} f_c(\log_b m) + \frac{c-b}{2b-2}\left(\frac{m}{b}\right)^{\log_b c-1} f_{c/b}(\log_b m) \\
&\quad + \left(\frac{m}{b}\right)^{\log_b c-1}\left[c\epsilon \log_b c f_c(\log_b m) + \frac{cb\epsilon}{\log b} f_c'(\log_b m)\right] + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big) \\
&= m^{\log_b c} f_c(\log_b m) + \frac{c-b}{2b-2}\frac{b}{c} m^{\log_b c-1} f_{c/b}(\log_b m) + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big) \\
&\qquad + \frac{b}{c} m^{\log_b c-1} \frac{c\epsilon}{\log b}\left[\log c f_c(\log_b m) + b f_c'(\log_b m)\right] \\
&= m^{\log_b c} f_c(\log_b m) + \frac{cb-b^2}{2bc-2c} m^{\log_b c-1} f_{c/b}(\log_b m) \\
&\qquad + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big) + \frac{b}{\log b}\left(1 - \left\{\frac{m}{b}\right\}\right) m^{\log_b c-1} \\
&\quad \times \left[\log c f_c(\log_b m) + b\left(-\log c f_c(\log_b m) + \frac{\log b}{c} f_{c/b}(\log_b m)\right)\right] \\
&= f_c(\log_b m)\ m^{\log_b c} + \frac{b(c-b)}{2c(b-1)} f_{c/b}(\log_b m)\ m^{\log_b c-1} \\
&\quad + b\left(1 - \left\{\frac{m}{b}\right\}\right)\left[(1-b)\log_b c f_c(\log_b m) + \frac{b}{c} f_{c/b}(\log_b m)\right] m^{\log_b c-1} \\
&\qquad + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big).
\end{aligned}
$$

So we see that the obstacle to having an asymptotic expansion à la Equation 2.17 is the rapidly fluctuating (on the scale of $\log_b m$) term $1 - \left\{\frac{m}{b}\right\}$. However, if we restrict the remainder of m modulo b, we see that this term is constant, and so we do obtain an asymptotic expansion of the desired type. Hence we obtain the following theorem:

Proposition 2.9. *Suppose that $b \mid c$. Then for any integer l, there are 1-periodic functions $f_{0,l}$ and $f_{1,l}$ such that for each k, $f_{k,l}$ is $1-k$ times differentiable and $f_{k,l}^{(1-k)}$ is Hölder continuous with exponent $\log_b\left(c/b\lceil c/b^2\rceil\right)$, and*

$$a_m = f_{0,l}(\log_b m) m^{\log_b c} + f_{1,l}(\log_b m) m^{\log_b c-1} + O\Big(m^{\log_b \lceil c/b^2 \rceil}\Big)$$

for $m \equiv l \mod b$.

It should be apparent from the above theorem that as far as the dependence of the functions $f_{k,l}$ on l is concerned, only the residue of l modulo b comes into play.

This theorem can be generalised further to obtain more precise asymptotic expansions, provided that more stringent conditions are imposed on c:

Theorem 2.10. *Suppose that $b^n \mid c$, where $n \in \mathbb{N}_0$. Then for any integer l modulo b^n, there are 1-periodic functions $f_{c,k,l}$ for $k \in \{0, 1, \dots, n\}$ such that for each k, $f_{c,k,l}$ is a linear combination of $f_c, f_{c/b}, \dots, f_{c/b^k}$ and*

$$a_m = f_{c,0,l}(\log_b m)m^{\log_b c} + f_{c,1,l}(\log_b m)m^{\log_b c-1} + \cdots + f_{c,n,l}(\log_b m)m^{\log_b c-n} \\ +O\left(m^{\log_b \lceil c/b^{n+1} \rceil}\right) \quad \text{for } m \equiv l \mod b^n.$$

Proof. It is apparent from the above asymptotic expansion that for different values of n, say n_1 and n_2, the functions $f_{c,k,l}$ appearing in the theorem for $n = n_1$ and those appearing for $n = n_2$ are the same, and so the $f_{c,k,l}$ may in fact be denoted without mention of the relevant value of n. Also, if $b^n \mid c$ we denote by $C_{c,n} = \left\langle f_c, f_{c/b}, \dots, f_{c/b^n} \right\rangle$ the set of functions which are linear combinations of $f_c, f_{c/b}, \dots, f_{c/b^n}$. If $n > 1$, $C_{c,n} \supset C_{c/b,n-1}$, and Corollary 2.7.1 states that if $s < n$, $f_c^{(s)} \in C_{c,s}$; in fact we see that if $f \in C_{c,n}$ and $b^{n+s} \mid c$ then $f^{(s)} \in C_{c,n+s}$.

We prove this theorem by induction on n. The case $n = 0$ is Theorem 2.6, and the case $n = 1$ is Proposition 2.9.

Now suppose that the theorem has been proved for $n - 1 \geq 0$, and that $b^n \mid c$ for some $c \in \mathbb{N}$. Let $l \in [0, b^n) \cap \mathbb{Z}$, and define $x_m = a_{b^n m+l}/(b^n m+l)^{\log_b c}$ and $M = b^n m + l$ for $m \in \mathbb{N}$. Then by Theorem 2.6,

$$\begin{aligned} x_m &= f_c(\log_b(b^n m + l)) + O\left((b^n m + l)^{-1}\right) \\ &= f_c(\log_b(m + l/b^n)) + O\left(m^{-1}\right) \\ &= f_c(\log_b m) + O\left(m^{-1}\right) \quad \text{for large } m; \text{ and so} \end{aligned}$$

$$x_m - x_{bm} \ll \frac{1}{m} \implies x_m = f_c(\log_b m) + \sum_{k=0}^{\infty} x_{b^k m} - x_{b^{k+1} m}. \tag{2.18}$$

Now letting $l_2 = \lfloor l/b \rfloor$ and $l_1 = l - bl_2$,

$$\begin{aligned} &x_m - x_{bm} \\ &= \frac{a_{b^n m+l}}{(b^n m + l)^{\log_b c}} - \frac{a_{b^{n+1} m+l}}{(b^{n+1} m + l)^{\log_b c}} \\ &= \frac{a_{b^n m+bl_2}}{(b^n m + l)^{\log_b c}} - \frac{a_{b(b^n m+l_2)}}{(b^{n+1} m + l)^{\log_b c}} \\ &= \frac{c a_{b^n m+bl_2}}{(b^{n+1} m + bl)^{\log_b c}} - \frac{a_{b^n m+l_2} + a_{b^n m+l_2-1} + \cdots + a_{b^n m+l_2-c+1}}{(b^{n+1} m + l)^{\log_b c}} \end{aligned}$$

$$\begin{aligned}&= ca_{b^n m+bl_2}\left(\frac{1}{(b^{n+1}m+bl)^{\log_b c}}-\frac{1}{(b^{n+1}m+l)^{\log_b c}}\right)\\&\quad+\frac{1}{(b^{n+1}m+l)^{\log_b c}}\big(ca_{b^n m+bl_2}-a_{b^n m+l_2}-a_{b^n m+l_2-1}-\cdots-a_{b^n m+l_2-c+1}\big).\end{aligned} \tag{2.20}$$

By the induction hypothesis, we have

$$\begin{aligned}&a_{b^n m+bl_2} = a_{b^n m+l} = a_M\\&= f_{c,0,l}(\log_b M)M^{\log_b c}+\cdots+f_{c,n-1,l}(\log_b M)M^{\log_b c-n+1}+O\big(M^{\log_b c-n}\big).\end{aligned}$$

where each $f_{c,k,l}\in C_{c,k}$. Also,

$$\begin{aligned}&\left(\frac{c}{(b^{n+1}m+bl)^{\log_b c}}-\frac{c}{(b^{n+1}m+l)^{\log_b c}}\right)\\&=\frac{1}{M^{\log_b c}}-\frac{1}{(M-(b-1)l/b)^{\log_b c}}=M^{-\log_b c}-\left(M-\frac{(b-1)l}{b}\right)^{-\log_b c}\\&= -\log_b c\frac{(b-1)l}{b}M^{-\log_b c-1}-\frac{(\log_b c)(\log_b c+1)}{2}\left(\frac{(b-1)l}{b}\right)^2M^{-\log_b c-2}\\&\qquad-\cdots-\binom{\log_b c+n-1}{n}\left(\frac{(b-1)l}{b}\right)^n M^{-\log_b c-n}+O\big(M^{-\log_b c-n-1}\big)\end{aligned}$$

by the binomial theorem. Hence by performing a Cauchy product, we have that

$$\begin{aligned}&ca_{b^n m+bl_2}\left(\frac{1}{(b^{n+1}m+bl)^{\log_b c}}-\frac{1}{(b^{n+1}m+l)^{\log_b c}}\right)\\&= g_0(\log_b M)M^{-1}+g_1(\log_b M)M^{-2}+\cdots+g_{n-1}(\log_b M)M^{-n}+O\big(M^{-n-1}\big)\end{aligned}$$

where each $g_k\in C_{c,k}$,

$$= \tilde{g}_0(\log_b m)m^{-1}+\tilde{g}_1(\log_b m)m^{-2}+\cdots+\tilde{g}_{n-1}(\log_b m)m^{-n}+O\big(m^{-n-1}\big) \tag{2.21}$$

where $\tilde{g}_k\in C_{c,k}$, by using Lemma A.1 and the binomial theorem on each term, and then collecting like terms in m.

Letting $l_2 = l_3 b + l_4$ where $l_4\in\{0,1,\ldots,b-1\}$ and noting that $b\mid c$, we also have that

$$\begin{aligned}
&ca_{b^n m+bl_2} - a_{b^n m+l_2} - a_{b^n m+l_2-1} - \cdots - a_{b^n m+l_2-c+1} \\
={}& ca_{b^n m+bl_2} - (l_4+1)a_{b^n m+l_3 b} - ba_{b^n m+(l_3-1)b} - \cdots - ba_{b^n m+(l_3-c/b+1)b} \\
&\qquad - (b-l_4-1)a_{b^n m+(l_3-c/b)b} \\
={}& c(a_{b^n m+bl_2} - a_{b^n m+(l_2-1)b}) + c(a_{b^n m+(l_2-1)b} - a_{b^n m+(l_2-2)b}) + \cdots \\
&+ c(a_{b^n m+(l_3+1)b} - a_{b^n m+l_3 b}) + (c-l_4-1)(a_{b^n m+l_3 b} - a_{b^n m+(l_3-1)b}) \\
&+ (c-b-l_4-1)(a_{b^n m+(l_3-1)b} - a_{b^n m+(l_3-2)b}) + \cdots \\
&+ (2b-l_4-1)(a_{b^n m+(l_3-c/b+2)b} - a_{b^n m+(l_3-c/b+1)b}) \\
&+ (b-l_4-1)(a_{b^n m+(l_3-c/b+1)b} - a_{b^n m+(l_3-c/b)b}) \\
={}& ca_{c,b^{n-1}m+l_2} + ca_{c,b^{n-1}m+l_2-1} + \cdots + ca_{c,b^{n-1}m+l_3+1} \\
&+ (c-l_4-1)a_{c,b^{n-1}m+l_3} + (c-b-l_4-1)a_{c,b^{n-1}m+l_3-1} + \cdots \\
&+ (2b-l_4-1)a_{c,b^{n-1}m+l_3-c/b+2} + (b-l_4-1)a_{c,b^{n-1}m+l_3-c/b+1} \\
={}& h_0(\log_b m)m^{\log_b c-1} + h_1(\log_b m)m^{\log_b c-2} + \cdots + h_{n-1}(\log_b m)m^{\log_b c-n} \\
&+O\left(m^{\log_b \lceil c/b^{n+1}\rceil}\right)
\end{aligned}$$

where each $h_k \in C_{c/b,k} \supset C_{c/k+1}$. This last expression was obtained by applying the induction hypothesis to each $a_{c,b^{n-1}m+r}$ term, this being valid since $b^{n-1} \mid \frac{c}{b}$, then using Lemma A.1 and the binomial theorem on each resultant term and collecting like terms in m. Also,

$$\begin{aligned}
&c^{n+1}(b^{n+1}m+l)^{-\log_b c} \\
={}& m^{-\log_b c} - \log_b c\left(\frac{l}{b^{n+1}}\right)m^{-\log_b c-1} + \binom{\log_b c+1}{2}\left(\frac{l}{b^{n+1}}\right)^2 m^{-\log_b c-2} \\
&+ \cdots + (-1)^{n-1}\binom{\log_b c+n-1}{n}\left(\frac{l}{b^{n+1}}\right)^{n-1} m^{-\log_b c-n+1} \\
&+O\left(m^{-\log_b c-n}\right).
\end{aligned}$$

Hence multiplying we have that

$$\begin{aligned}
&\frac{1}{(b^{n+1}m+l)^{\log_b c}}(ca_{b^n m+bl_2} - a_{b^n m+l_2} - a_{b^n m+l_2-1} - \cdots - a_{b^n m+l_2-c+1}) \\
={}& \tilde{h}_0(\log_b m)m^{-1} + \tilde{h}_1(\log_b m)m^{-2} + \cdots + \tilde{h}_{n-1}(\log_b m)m^{-n} \\
&+O\left(m^{\log_b \lceil c/b^{n+1}\rceil/c}\right)
\end{aligned} \tag{2.22}$$

where each $\tilde{h}_k \in C_{c/b,k} \supset C_{c,k+1}$.

Combining Equation 2.20, Equation 2.21 and Equation 2.22 then yields

$$x_m - x_{bm} = \tilde{f}_0(\log_b m)m^{-1} + \tilde{f}_1(\log_b m)m^{-2} + \cdots + \tilde{f}_{n-1}(\log_b m)m^{-n} + O\Big(m^{\log_b \lceil c/b^{n+1}\rceil /c}\Big)$$

where each $\tilde{f}_k \in C_{c,k+1}$, giving

$$\begin{aligned} & x_{b^k m} - x_{b^{k+1}m} \\ &= \tilde{f}_0(\log_b m)b^{-k}m^{-1} + \tilde{f}_1(\log_b m)b^{-2k}m^{-2} + \cdots + \tilde{f}_{n-1}(\log_b m)b^{-nk}m^{-n} \\ &\qquad + O\Big(b^{k\log_b \lceil c/b^{n+1}\rceil /c} m^{\log_b \lceil c/b^{n+1}\rceil /c}\Big). \end{aligned}$$

Thus Equation 2.18 gives

$$\begin{aligned} & x_m \\ &= f_c(\log_b m) + \sum_{k=0}^{\infty} x_{b^k m} - x_{b^{k+1}m} \\ &= f_c(\log_b m) + \tilde{f}_0(\log_b m)\sum_{k=0}^{\infty} b^{-k}m^{-1} + \tilde{f}_1(\log_b m)\sum_{k=0}^{\infty} b^{-2k}m^{-2} + \cdots \\ &\quad + \tilde{f}_{n-1}(\log_b m)\sum_{k=0}^{\infty} b^{-nk}m^{-n} + O\left(\sum_{k=0}^{\infty} b^{k\log_b \lceil c/b^{n+1}\rceil /c} m^{\log_b \lceil c/b^{n+1}\rceil /c}\right) \\ &= f_c(\log_b m) + \frac{\tilde{f}_0(\log_b m)}{1-b^{-1}}m^{-1} + \frac{\tilde{f}_1(\log_b m)}{1-b^{-2}}m^{-2} + \cdots \\ &\qquad + \frac{\tilde{f}_{n-1}(\log_b m)}{1-b^{-n}}m^{-n} + O\left(\frac{m^{\log_b \lceil c/b^{n+1}\rceil /c}}{1-b^{-\log_b \lceil c/b^{n+1}\rceil /c}}\right) \\ &= \hat{f}_0(\log_b m) + \hat{f}_1(\log_b m)m^{-1} + \cdots + \hat{f}_n(\log_b m)m^{-n} + O\Big(m^{\log_b \lceil c/b^{n+1}\rceil /c}\Big) \end{aligned}$$

where each $\hat{f}_k \in C_{c,k}$. And so finally

$$\begin{aligned} a_M &= x_m \cdot M^{\log_b c} \\ &= M^{\log_b c} \times \left[\hat{f}_0\left(\log_b \frac{M-l}{b^n}\right) + \hat{f}_1\left(\log_b \frac{M-l}{b^n}\right)\frac{b^n}{M-l} + \cdots \right. \\ &\qquad \left. + \hat{f}_n\left(\log_b \frac{M-l}{b^n}\right)\left(\frac{b^n}{M-l}\right)^n + O\left(\left(\frac{M-l}{b^n}\right)^{\log_b \lceil c/b^{n+1}\rceil /c}\right)\right] \\ &= M^{\log_b c} \times \left[\hat{f}_0(\log_b(M-l)) + \hat{f}_1(\log_b(M-l))\frac{b^n}{M-l} + \cdots \right. \\ &\qquad \left. + \hat{f}_n(\log_b(M-l))\left(\frac{b^n}{M-l}\right)^n + O\Big(M^{\log_b \lceil c/b^{n+1}\rceil /c}\Big)\right] \end{aligned}$$

$$= \grave{f}_0(\log_b M)M^{\log_b c} + \grave{f}_1(\log_b M)M^{\log_b c-1} + \cdots + \grave{f}_n(\log_b M)M^{\log_b c-n} \\ +O\left(M^{\log_b \lceil c/b^{n+1}\rceil/c}\right)$$

where each $\grave{f}_k \in C_{c,k}$, by applying the binomial theorem to each $\left(\frac{b^n}{M-l}\right)^k$ term and Lemma A.1 to each $\hat{f}_k(\log_b(M-l))$ term, and then gathering like powers of M. So the induction step is finished, and the theorem is proved. □

Now the $f_{c,k,l}$ are defined uniquely by the above theorem, and it is apparent on comparison with Theorem 2.6 that when $k = 0$, the $f_{c,0,l}$ above is in fact simply the f_c defined earlier, and so in fact has no dependence on l. The following proposition collects similar properties of the $f_{c,k,l}$ as well as a relation which will allow the $f_{c,k,l}$ for $k > 0$ to be computed recursively in terms of the f_c.

Proposition 2.11. *Let $l, c \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that $b^n \mid c$. Then the functions $f_{c,k,l}$ defined in Theorem 2.10 satisfy the following properties:*

(a) If $n > 0$, the $f_{c,k,l}$ defined in the use of Theorem 2.10 for n and those in the use of Theorem 2.10 for $n-1$ are the same, and hence may be denoted without reference to n.

(b) $f_{c,0,l} = f_c$.

(c) For each $k \in \{0, 1, \ldots, n\}$, $f_{c,k,l}$ only depends on l modulo b^k, i.e.

$$f_{c,k,l+eb^k} = f_{c,k,l} \quad \textit{for any } e \in \mathbb{Z}.$$

(d) If $n > 0$ and $r \in \{0, 1, \ldots, b\}$,

$$f_{c,n,bl+r} = \sum_{k=1}^{n-1}\sum_{i=0}^{n-k}\sum_{s=0}^{i} \frac{(-1)^{n+k+i+s+1}}{s!(\log b)^s}\binom{\log_b c - k}{n-k-i}\alpha_{s,i} \\ \times \left[\frac{b^k}{c}\sum_{h=0}^{b^{n-1}-1}(bh+r)^{n-k} f^{(s)}_{c/b,k-1,l-h} + b^{n(n-k)} f^{(s)}_{c,k,bl+r}\right] \\ - \frac{b^n}{c}\sum_{h=0}^{b^{n-1}-1} f_{c/b,n-1,l-h} - \sum_{i=0}^{n}\sum_{s=0}^{i}\binom{\log_b c}{n-i}\frac{(-1)^{s+n+i}}{s!(\log b)^s} b^{n^2}\alpha_{s,i} f^{(s)}_c$$

where for each $s \in \mathbb{N}_0$,

$$\alpha_{s,i} = [x^i]\left(\log\left(\frac{1}{1-x}\right)\right)^s,$$

or equivalently

$$\left(\log\left(\frac{1}{1-x}\right)\right)^s = \sum_{i=0}^{\infty} \alpha_{s,i} x^i = \sum_{i=s}^{\infty} \alpha_{s,i} x^i.$$

We see that in property (d) above, $f_{c,n,bl+r}$ is written as a linear combination of $f_{c,k,t}$ and $f_{c/b,k,t}$ where $0 \le k < n$ and $t \in \mathbb{Z}$. Thus we can use this property together with the fact that $f_{c,0,l} = f_c$ for $l \in \mathbb{Z}$ to recursively compute $f_{c,k,l}$ in terms of the f_c, say using a computer algebra system.

Proof. The first two properties given above are easy to show, and as such are only listed for completeness. Here we continue with the proofs of the consequent properties:

(c) By Theorem 2.10 applied for $n = k$, since $b^k \mid b^n \mid c$,

$$a_m = f_{c,0,l}(\log_b m) m^{\log_b c} + \cdots + f_{c,k,l}(\log_b m) m^{\log_b c-k} + O\big(m^{\log_b c-k-1}\big),$$

valid for large $m \equiv l \mod b^k$, and here $f_{c,k,l}$ depends on l modulo b^k. By property (a), the $f_{c,k,l}$ appearing in the application of Theorem 2.10 for n is the same as that when $n = k$, and so we are done.

(d) Let $m \equiv l \mod b^n$ be large, and let $M = bm + r$. Then by Theorem 2.10,

$$\begin{aligned} &a_M - a_{M-b^n} \\ &= \sum_{k=0}^{n} f_{c,k,bl+r}(\log_b M) M^{\log_b c-k} - \sum_{k=0}^{n} f_{c,k,bl+r}(\log_b(M-b^n))(M-b^n)^{\log_b c-k} \\ &\qquad + O\Big(M^{\log_b \lceil c/b^{n+1}\rceil}\Big). \end{aligned}$$

Now by Lemma A.1, for each $k \in \{0, 1, \ldots, n\}$ we have the expansion

$$\begin{aligned} &f_{c,k,bl+r}(\log_b(M-b^n)) = f_{c,k,bl+r}(\log_b M + \log_b(1 - b^n/M)) \\ &= \sum_{s=0}^{n-k} \frac{f^{(s)}_{c,k,bl+r}(\log_b M)}{s!} (\log_b(1-b^n/M))^s + O\Big((b^n/M)^{\log_b c/\lceil c/b^{n+1}\rceil - k}\Big) \\ &= \sum_{s=0}^{n-k} \frac{f^{(s)}_{c,k,bl+r}(\log_b M)}{s!} \left(\frac{-1}{\log b}\right)^s \sum_{i=s}^{\infty} \alpha_{s,i} \left(\frac{b^n}{M}\right)^i + O\Big(M^{k+\log_b \lceil c/b^{n+1}\rceil/c}\Big) \\ &= \sum_{s=0}^{n-k} \frac{(-1)^s}{s!(\log b)^s} f^{(s)}_{c,k,bl+r}(\log_b M) \sum_{i=s}^{n-k} \alpha_{s,i} b^{ni} M^{-i} + O\Big(M^{k+\log_b \lceil c/b^{n+1}\rceil/c}\Big) \\ &= \sum_{i=0}^{n-k} M^{-i} \sum_{s=0}^{i} \frac{(-1)^s}{s!(\log b)^s} f^{(s)}_{c,k,bl+r}(\log_b M) \alpha_{s,i} b^{ni} + O\Big(M^{k+\log_b \lceil c/b^{n+1}\rceil/c}\Big). \end{aligned}$$

Also, by the binomial theorem we have

$$\begin{aligned}(M-b^n)^{\log_b c-k} &= M^{\log_b c-k}\left(1-\frac{b^n}{M}\right)^{\log_b c-k}\\ &= M^{\log_b c-k}\sum_{j=0}^{\infty}(-1)^j\binom{\log_b c-k}{j}\left(\frac{b^n}{M}\right)^j\\ &= \sum_{j=0}^{n-k}(-1)^j\binom{\log_b c-k}{j}b^{nj}M^{\log_b c-k-j}+O\big(M^{\log_b c-n-1}\big).\end{aligned}$$

Thus with $x_M=\log_b M$ for each M, a_{M-b^n}

$$\begin{aligned}&=\sum_{k=0}^{n} f_{c,k,bl+r}(\log_b(M-b^n))(M-b^n)^{\log_b c-k}+O\Big(M^{\log_b\lceil c/b^{n+1}\rceil}\Big)\\
&=\sum_{k=0}^{n}\left[\sum_{i=0}^{n-k}M^{-i}\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}f^{(s)}_{c,k,bl+r}(x_M)\alpha_{s,i}b^{ni}+O\Big(M^{k+\log_b\lceil c/b^{n+1}\rceil/c}\Big)\right]\\
&\qquad\times\left[\sum_{j=0}^{n-k}(-1)^j\binom{\log_b c-k}{j}b^{nj}M^{\log_b c-k-j}+O\big(M^{\log_b c-n-1}\big)\right]\\
&\qquad\qquad+O\Big(M^{\log_b\lceil c/b^{n+1}\rceil}\Big)\\
&=\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{n-k}M^{\log_b c-k-i-j}(-1)^j\binom{\log_b c-k}{j}b^{n(i+j)}\\
&\qquad\times\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}f^{(s)}_{c,k,bl+r}(x_M)\alpha_{s,i}+O\Big(M^{\log_b\lceil c/b^{n+1}\rceil}\Big)\\
&=\sum_{k=0}^{n}\sum_{j=0}^{n-k}M^{\log_b c-k-j}\sum_{i=0}^{j}(-1)^{j-i}b^{nj}\binom{\log_b c-k}{j-i}\\
&\qquad\times\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}\alpha_{s,i}f^{(s)}_{c,k,bl+r}(x_M)+O\Big(M^{\log_b\lceil c/b^{n+1}\rceil}\Big)\\
&=\sum_{j=0}^{n}M^{\log_b c-j}\sum_{k=0}^{j}\sum_{i=0}^{j-k}\sum_{s=0}^{i}\binom{\log_b c-k}{j-k-i}\frac{(-1)^{s+j-k-i}b^{n(j-k)}}{s!(\log b)^s}\alpha_{s,i}f^{(s)}_{c,k,bl+r}(x_M)\\
&\qquad\qquad+O\Big(M^{\log_b\lceil c/b^{n+1}\rceil}\Big).\end{aligned}$$

Thus $a_M - a_{M-b^n}$

$$= \sum_{j=1}^{n} M^{\log_b c - j} \sum_{k=0}^{j}\sum_{i=0}^{j-k}\sum_{s=0}^{i} \binom{\log_b c - k}{j-k-i} \frac{(-1)^{s+j+k+i-1}}{s!(\log b)^s} b^{n(j-k)} \alpha_{s,i} f^{(s)}_{c,k,bl+r}(x_M) + O\Big(M^{\log_b \lceil c/b^{n+1} \rceil}\Big). \tag{2.23}$$

On the other hand,

$$\begin{aligned}
&a_M - a_{M-b^n} = a_{bm} - a_{b(m-b^{n-1})} \\
&= a_{c,m} + a_{c,m-1} + \cdots + a_{c,m-b^{n-1}+1} = \sum_{h=0}^{b^{n-1}-1} a_{c,m-h} \\
&= \sum_{h=0}^{b^{n-1}-1}\sum_{k=0}^{n-1} (m-h)^{\log_b (c/b) - k} f_{c/b,k,l-h}(\log_b(m-h)) + O\Big(m^{\log_b \lceil (c/b)/b^{n-1+1} \rceil}\Big) \\
&= \sum_{h=0}^{b^{n-1}-1}\sum_{k=0}^{n-1} \left(\frac{M-r-bh}{b}\right)^{\log_b c - k - 1} f_{c/b,k,l-h}(\log_b(M-r-bh)) \\
&\qquad + O\Big(M^{\log_b \lceil c/b^{n+1} \rceil}\Big)
\end{aligned}$$

since $m - h = \frac{M-r}{b} - h = \frac{M-r-bh}{b}$ and $f_{c/b,k,l-h}$ is 1-periodic.

Similarly to before, with $\tilde{c} = c/b$ and $x_M = \log_b M$ we have for each h and k the 'Taylor' expansion

$$\begin{aligned}
&f_{\tilde{c},k,l-h}(\log_b(M-r-bh)) = f_{\tilde{c},k,l-h}\left(\log_b M + \log_b\left(1 - \frac{bh+r}{M}\right)\right) \\
&= \sum_{s=0}^{n-1-k} \frac{f^{(s)}_{\tilde{c},k,l-h}(\log_b M)}{s!} \left(\log_b\left(1 - \frac{bh+r}{M}\right)\right)^s + O\left(\left(\frac{bh+r}{M}\right)^{\log_b \tilde{c}/\lceil \tilde{c}/b^n \rceil - k}\right) \\
&= \sum_{s=0}^{n-1-k} \frac{f^{(s)}_{\tilde{c},k,l-h}(\log_b M)}{s!} \left(\frac{-1}{\log b}\right)^s \sum_{i=s}^{\infty} \alpha_{s,i} \left(\frac{bh+r}{M}\right)^i + O\Big(M^{k+1+\log_b \lceil c/b^{n+1} \rceil / c}\Big) \\
&= \sum_{s=0}^{n-1-k} \frac{(-1)^s}{s!(\log b)^s} f^{(s)}_{\tilde{c},k,l-h}(x_M) \sum_{i=s}^{n-1-k} \alpha_{s,i}(bh+r)^i M^{-i} + O\Big(M^{k+1+\log_b \lceil c/b^{n+1} \rceil / c}\Big) \\
&= \sum_{i=0}^{n-1-k} M^{-i}(bh+r)^i \sum_{s=0}^{i} \frac{(-1)^s}{s!(\log b)^s} \alpha_{s,i} f^{(s)}_{\tilde{c},k,l-h}(x_M) + O\Big(M^{k+1+\log_b \lceil c/b^{n+1} \rceil / c}\Big).
\end{aligned}$$

Also, by the binomial theorem we have

$$\left(\frac{M-r-bh}{b}\right)^{\log_b c-k-1} = \frac{M^{\log_b c-k-1}}{c\cdot b^{-k-1}}\left(1-\frac{bh+r}{M}\right)^{\log_b c-k-1}$$
$$= \frac{b^{k+1}}{c}M^{\log_b c-k-1}\sum_{j=0}^{\infty}(-1)^j\binom{\log_b c-k-1}{j}\left(\frac{bh+r}{M}\right)^j$$
$$= \frac{b^{k+1}}{c}\sum_{j=0}^{n-1-k}(-1)^j\binom{\log_b c-k-1}{j}(bh+r)^j M^{\log_b c-k-1-j}+O\bigl(M^{\log_b c-n-1}\bigr).$$

Thus for each $k\in\{0,1,\ldots,n-1\}$,

$$\sum_{h=0}^{b^{n-1}-1}\left(\frac{M-r-bh}{b}\right)^{\log_b c-k-1} f_{\tilde{c},k,l-h}(\log_b(M-r-bh))$$
$$= \frac{b^{k+1}}{c}\sum_{h=0}^{b^{n-1}-1}$$
$$\left[\sum_{j=0}^{n-1-k}M^{\log_b c-k-1-j}(-1)^j\binom{\log_b c-k-1}{j}(bh+r)^j+O\bigl(M^{\log_b c-n-1}\bigr)\right]$$
$$\times\left[\sum_{i=0}^{n-1-k}M^{-i}(bh+r)^i\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}\alpha_{s,i}f^{(s)}_{\tilde{c},k,l-h}(x_M)+O\Bigl(M^{k+1+\log_b\lceil c/b^{n+1}\rceil/c}\Bigr)\right]$$
$$= \frac{b^{k+1}}{c}\sum_{h=0}^{b^{n-1}-1}\sum_{j=0}^{n-1-k}M^{\log_b c-k-1-j}(bh+r)^j\sum_{i=0}^{j}(-1)^{j-i}\binom{\log_b c-k-1}{j-i}$$
$$\times\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}\alpha_{s,i}f^{(s)}_{\tilde{c},k,l-h}(x_M)+O\Bigl(M^{\log_b\lceil c/b^{n+1}\rceil}\Bigr)$$
$$= \frac{b^{k+1}}{c}\sum_{j=0}^{n-1-k}M^{\log_b c-k-j-1}\sum_{i=0}^{j}(-1)^{j-i}\binom{\log_b c-k-1}{j-i}$$
$$\times\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}\alpha_{s,i}\sum_{h=0}^{b^{n-1}-1}(bh+r)^j f^{(s)}_{\tilde{c},k,l-h}(x_M)+O\Bigl(M^{\log_b\lceil c/b^{n+1}\rceil}\Bigr).$$

Thus $a_M-a_{M-b^n}$

$$= \sum_{k=1}^{n}\frac{b^k}{c}\sum_{j=0}^{n-k}M^{\log_b c-k-j}\sum_{i=0}^{j}(-1)^{j-i}\binom{\log_b c-k}{j-i}$$
$$\times\sum_{s=0}^{i}\frac{(-1)^s}{s!(\log b)^s}\alpha_{s,i}\sum_{h=0}^{b^{n-1}-1}(bh+r)^j f^{(s)}_{\tilde{c},k-1,l-h}(x_M)+O\Bigl(M^{\log_b\lceil c/b^{n+1}\rceil}\Bigr)$$

$$
\begin{aligned}
= \sum_{j=1}^{n} M^{\log_b c - j} & \sum_{k=1}^{j} \sum_{i=0}^{j-k} \sum_{s=0}^{i} \frac{b^k}{c} \frac{(-1)^{j-k-i+s}}{s!(\log b)^s} \binom{\log_b c - k}{j-k-i} \alpha_{s,i} \\
& \times \sum_{h=0}^{b^{n-1}-1} (bh+r)^{j-k} f^{(s)}_{\tilde{c},k-1,l-h}(x_M) + O\Big(M^{\log_b \lceil c/b^{n+1} \rceil}\Big).
\end{aligned} \tag{2.24}
$$

Comparing Equation 2.23 and Equation 2.24, since the 'coefficients' of each of the powers of M are continuous 1-periodic functions of $\log_b M$, the respective 'coefficients' are the same. In particular, the 'coefficients' of $M^{\log_b c - n}$ are the same and so

$$
\begin{aligned}
& \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{s=0}^{i} \binom{\log_b c - k}{n-k-i} \frac{(-1)^{s+n+k+i+1}}{s!(\log b)^s} b^{n(n-k)} \alpha_{s,i} f^{(s)}_{c,k,bl+r} \\
= & \sum_{k=1}^{n} \sum_{i=0}^{n-k} \sum_{s=0}^{i} \frac{b^k}{c} \frac{(-1)^{n+k+i+s}}{s!(\log b)^s} \binom{\log_b c - k}{n-k-i} \alpha_{s,i} \\
& \qquad \times \sum_{h=0}^{b^{n-1}-1} (bh+r)^{n-k} f^{(s)}_{c/b,k-1,l-h} \\
\implies f_{c,n,bl+r} = & \sum_{k=1}^{n-1} \sum_{i=0}^{n-k} \sum_{s=0}^{i} \frac{(-1)^{n+k+i+s+1}}{s!(\log b)^s} \binom{\log_b c - k}{n-k-i} \alpha_{s,i} \\
& \qquad \times \left[\frac{b^k}{c} \sum_{h=0}^{b^{n-1}-1} (bh+r)^{n-k} f^{(s)}_{c/b,k-1,l-h} + b^{n(n-k)} f^{(s)}_{c,k,bl+r} \right] \\
& - \frac{b^n}{c} \sum_{h=0}^{b^{n-1}-1} f_{c/b,n-1,l-h} - \sum_{i=0}^{n} \sum_{s=0}^{i} \binom{\log_b c}{n-i} \frac{(-1)^{s+n+i}}{s!(\log b)^s} b^{n^2} \alpha_{s,i} f^{(s)}_{c}
\end{aligned}
$$

since $f_{c,0,l} = f_c$ for any l by property (b). $\square$

Bibliography

[1] Jean-Paul Allouche and Jeffrey Shallit. "The ring of k-regular sequences. II". In: *Theoretical Computer Science* 307.1 (2003), pp. 3–29.

[2] Jean-Paul Allouche and Jeffrey Shallit. "The ring of k-regular sequences". In: *Theoretical Computer Science* 98.2 (1992), pp. 163–197.

[3] Katherine Anders et al. "Congruence Properties of Binary Partition Functions". In: *Annals of Combinatorics* 17.1 (2013), pp. 15–26.

[4] George E Andrews. "Congruence properties of the m-ary partition function". In: *Journal of Number Theory* 3.1 (1971), pp. 104–110.

[5] N. G. de Bruijn. "On Mahler's partition problem". In: *Nederl. Akad. Wetensch., Proc.* 51 (1948). = Indagationes Math. 10, 210–220 (1948), pp. 659–669.

[6] R. F. Churchhouse. "Congruence properties of the binary partition function". In: *Proc. Cambridge Philos. Soc.* 66 (1969), pp. 371–376.

[7] Gunnar Dirdal. "Congruence properties for a class of arithmetical functions." In: *Mathematica Scandinavica* 38.0 (1976), pp. 247–261.

[8] Gunnar Dirdal. "Congruences for m-ary partitions." In: *Mathematica Scandinavica* 37.0 (1975), pp. 76–82.

[9] Gunnar Dirdal. "On restricted m-ary partitions." In: *Mathematica Scandinavica* 37.0 (1975), pp. 51–60.

[10] De-Jun Feng, Pierre Liardet, and Alain Thomas. "Partition functions in numeration systems with bounded multiplicity". In: *Unif. Distrib. Theory* 9.1 (2014), pp. 43–77.

[11] Hansraj Gupta. "Proof of the Churchhouse conjecture concerning binary partitions". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 70 (01 July 1971), pp. 53–56.

[12] Michael D Hirschhorn, James A Sellers, et al. "A different view of m-ary partitions". In: *Australasian Journal of Combinatorics* 30 (2004), pp. 193–196.

[13] Donald E. Knuth. “An almost linear recurrence”. In: *Fibonacci Quart.* 4 (1966), pp. 117–128.

[14] Daniel Krenn, Dimbinaina Ralaivaosaona, and Stephan Wagner. “On the number of multi-base representations of an integer”. In: *Proceedings of the 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*. Ed. by Mireille Bousquet-Mélou and Michèle Soria. DMTCS-HAL Proceedings Series. Paris, June 2014, pp. 229–240.

[15] Kurt Mahler. “On a special functional equation”. In: *J. London Math. Soc.* 15 (1940), pp. 115–123.

[16] Carl Meyer. *Matrix analysis and applied linear algebra*. With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.) Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, pp. xii+718. ISBN: 0-89871-454-0.

[17] W. B. Pennington. “On Mahler’s Partition Problem”. In: *Annals of Mathematics*. Second Series 57.3 (1953), pp. 531–546.

[18] Vladimir Yu. Protasov. “Asymptotic behaviour of the partition function”. In: *Sbornik: Mathematics* 191.3 (2000), p. 381.

[19] Vladimir Yu. Protasov. “On the problem of the asymptotics of the partition function”. In: *Mat. Zametki* 76.1 (2004), pp. 151–156.

[20] Øystein Rødseth. “Some arithmetical properties of m-ary partitions”. In: *Math. Proc. Cambridge Philos. Soc.* 68 (02 Sept. 1970), pp. 447–453.

[21] Øystein Rødseth and James A Sellers. “On a restricted m-non-squashing partition function”. In: *Journal of Integer Sequences* 8.2 (2005), p. 3.

[22] Øystein Rødseth and James A Sellers. “On m-ary Partition Function Congruences: A Fresh Look at a Past Problem”. In: *Journal of Number Theory* 87.2 (2001), pp. 270–281.

[23] Bruce Reznick. “Some binary partition functions”. In: *Analytic number theory (Allerton Park, IL, 1989)*. Vol. 85. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 451–477.

Appendix A

Hölder continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a Hölder condition, or to be Hölder continuous, if there is a nonnegative constant α such that

$$f(x) - f(y) \ll |x - y|^{\alpha}.$$

The number α above is called the *exponent* of the Hölder condition, and the function f can be said to be Hölder continuous with exponent α, or simply α-Hölder continuous. Hölder continuity can be formulated analogously for functions between any two metric spaces, but in this thesis we will focus on the case of real-valued periodic functions.

The following lemma will be used in the thesis:

Lemma A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be n times continuously differentiable where $n \in \mathbb{N}_0$, and suppose that $f^{(n)}$ is Hölder continuous of exponent $r \in (0, 1]$. Then for $x, y \in \mathbb{R}$ with $y - x$ small,*

$$f(y) = f(x)+f'(x)(y-x)+\frac{f''(x)}{2!}(y-x)^2+\cdots+\frac{f^{(n)}(x)}{n!}(y-x)^n+O\big(|y - x|^{n+r}\big)$$

Proof. We proceed by induction on n. The case $n = 0$ is equivalent to

$$f(y) - f(x) = O(|y - x|^r),$$

which follows from the Hölder continuity of $f = f^{(0)}$.

Now suppose that the lemma has been proved for some $n \in \mathbb{N}_0$, and let $f : \mathbb{R} \to \mathbb{R}$ be $n+1$ times continuously differentiable with $f^{(n+1)}$ being Hölder continuous with exponent r. Then the induction hypothesis applies to f', and so

$$\begin{aligned}
f(y) &= f(x) + \int_x^y f'(t)\,\mathrm{d}t \quad \text{by the Fundamental Theorem of Calculus} \\
&= f(x) + \int_x^y \sum_{k=1}^{n+1} \frac{f^{(k)}(x)}{(k-1)!}(t-x)^{k-1} + O\big(|t-x|^{n+r}\big)\,\mathrm{d}t \\
&= f(x) + \sum_{k=1}^{n+1} \frac{f^{(k)}(x)}{k!}(y-x)^k + O\left(\int_x^y |t-x|^{n+r}\,\mathrm{d}t\right) \\
&= f(x) + f'(x)(y-x) + \cdots + \frac{f^{(n+1)}(x)}{(n+1)!}(y-x)^{n+1} + O\big(|y-x|^{n+1+r}\big).
\end{aligned}$$

This completes the induction step, and so the lemma is proved. $\square$

Appendix B

Proofs of Auxiliary Constructive Theorems

In this section, we give the proofs of two theorems used in Chapter 2 which each show that given certain analytic conditions on a sequence x_m, the sequence is asymptotically log-periodic:

Lemma 2.1. *Let $b \in \mathbb{N}$ be greater than 1, let $\eta > 0$ and let $(x_m)_{m\in\mathbb{N}}$ be a sequence of real numbers such that $x_m - x_{bm} \ll m^{-\eta}$ and $x_{m+1} - x_m \ll m^{-\eta}$ for large m. Then there is a 1-periodic continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $x_m = f(\log_b m) + O(m^{-\eta})$ for large m. Moreover, if $\eta \leq 1$, then the function f is Hölder continuous with exponent η, and if $\eta > 1$ then f is constant. In particular, if $\eta = 1$ then f is Lipschitz continuous.*

Proof. We define the continuous function g on $[0, \infty)$ by $g(\log_b m) = x_m$ for each $m \in \mathbb{N}$ and by linear interpolation between each two consecutive points $\log_b m$ and $\log_b(m + 1)$, i.e. for each $m \in \mathbb{N}$ and $t \in [0, 1]$,

$$\begin{aligned} g(t \log_b m + (1 - t) \log_b(m + 1)) &= tg(\log_b m) + (1 - t)g(\log_b(m + 1)) \\ &= tx_m + (1 - t)x_{m+1}. \end{aligned}$$

Now

$$g(1 + \log_b m) = g(\log_b(bm)) = x_{bm} = x_m + O\big(m^{-\eta}\big) = g(\log_b m) + O\big(m^{-\eta}\big)$$

for any $m \in \mathbb{N}$. If $x \in [0, \infty)$, we know that $\log_b m \leq x < \log_b(m+1)$ for some unique $m \in \mathbb{N}$. Similarly, $\log_b n \leq 1 + x < \log_b(n + 1)$ for some unique $n \in \mathbb{N}$. Hence $\log_b(bm) \leq 1 + x < \log_b(bm + b)$, and so $n \geq bm$ and $n + 1 \leq bm + b$, or $bm \leq n < bm + b$.

Now by the linear interpolation definition of g,

$$\begin{aligned} g(x) &= g(\log_b m) + \frac{x - \log_b m}{\log_b(m+1)/m}\left(g(\log_b(m+1)) - g(\log_b m)\right) \\ &= x_m + O\left(x_{m+1} - x_m\right) \\ &= x_m + O\left(m^{-\eta}\right), \end{aligned}$$

and similarly

$$\begin{aligned} g(1+x) &= x_n + O\left(n^{-\eta}\right) \\ &= x_{bm} + (x_n - x_{n-1}) + \cdots + (x_{bm+1} - x_{bm}) + O\left((bm)^{-\eta}\right) \\ &= x_{bm} + O\left((bm)^{-\eta}\right). \end{aligned}$$

Thus

$$\begin{aligned} g(1+x) = x_{bm} + O\left(m^{-\eta}\right) &= x_m + O\left(m^{-\eta}\right) \\ &= g(x) + O\left(m^{-\eta}\right) = g(x) + O\left(b^{-\eta\log_b m}\right) \\ &= g(x) + O\left(b^{-\eta x}\right), \end{aligned}$$

and so

$$\sum_{k=0}^{\infty} g(x+k+1) - g(x+k) \ll \sum_{k=0}^{\infty} b^{-\eta(x+k)} = \frac{b^{-\eta x}}{1-b^{-\eta}} \ll b^{-\eta x}.$$

Thus $f(x) := \lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} g(x+k)$ exists for each $x \in \mathbb{R}$. This function f is evidently 1-periodic, and moreover $f(x) - g(x) \ll b^{-\eta x}$ – in particular,

$$x_m = g(\log_b m) = f(\log_b m) + O\left(b^{-\eta\log_b m}\right) = f(\log_b m) + O\left(m^{-\eta}\right).$$

Now define $\varepsilon = b^{-\eta}$ and consider the function $h : [0,\infty) \to \mathbb{R}$ defined piecewise on each interval $[\log_b m, \log_b(m+1))$ by

$$h(x) = \frac{x_{m+1} - x_m}{\log_b(m+1) - \log_b m} \ll \frac{m^{-\eta}}{1/m} = m^{\log_b(b\varepsilon)} = (b\varepsilon)^{\log_b m} \ll (b\varepsilon)^x.$$

Moreover, $\int_a^b h(t)\mathrm{d}t = g(b) - g(a)$ for each $a, b \geq 0$. Now suppose that $\varepsilon > 1/b$, and let $0 \leq x, y < 2$. Then for each $n \in \mathbb{N}$,

$$g(y+n) - g(x+n) = \int_{x+n}^{y+n} h(t)\mathrm{d}t \ll \int_{x+n}^{y+n} (b\varepsilon)^t \mathrm{d}t \ll \left|(b\varepsilon)^y - (b\varepsilon)^x\right| (b\varepsilon)^n.$$

Also, $f(y) = f(y+n) = g(y+n) + O\big(b^{-\eta(y+n)}\big) = g(y+n) + O(\varepsilon^{y+n})$ and similarly $f(x) = g(x+n) + O\left(\varepsilon^{x+n}\right)$. Hence

$$\begin{aligned} f(y) - f(x) &= g(y+n) - g(x+n) + O\left(\varepsilon^{y+n} + \varepsilon^{x+n}\right) \\ &\ll \left|(b\varepsilon)^y - (b\varepsilon)^x\right| (b\varepsilon)^n + (\varepsilon^x + \varepsilon^y)\varepsilon^n \\ &\ll -\log_b \varepsilon \times \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| (b\varepsilon)^n + \log_b(b\varepsilon) \times \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\varepsilon^n. \end{aligned} \tag{B.1}$$

Now by the (weighted) arithmetic mean - geometric mean inequality,

$$\begin{aligned} -\log_b \varepsilon \times \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| (b\varepsilon)^n + \log_b(b\varepsilon) \times \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\varepsilon^n \\ \geq \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right|^{-\log_b \varepsilon} \left(\frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\right)^{\log_b(b\varepsilon)} \qquad (=: L) \end{aligned}$$

since $\log_b \varepsilon \in (-1, 0)$, and moreover equality occurs if and only if

$$\begin{aligned} \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| (b\varepsilon)^n &= \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\varepsilon^n \\ \iff b^n &= \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2} \Bigg/ \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| \\ \iff n = n_c &:= \log_b \left(\frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2} \Bigg/ \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right|\right). \end{aligned}$$

Note that since $0 \leq x, y < 2$, $n_c > 0$. Hence we may put $n = \lceil n_c \rceil$ in Equation B.1 to obtain

$$\begin{aligned} f(y) - f(x) &\ll -\log_b \varepsilon \times \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| (b\varepsilon)^{\lceil n_c \rceil} + \log_b(b\varepsilon) \times \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\varepsilon^{\lceil n_c \rceil} \\ &< (b\varepsilon) \times \left(-\log_b \varepsilon \times \left|\frac{(b\varepsilon)^y - (b\varepsilon)^x}{(b\varepsilon)^2 - 1}\right| (b\varepsilon)^{n_c} + \log_b(b\varepsilon) \times \frac{\varepsilon^x + \varepsilon^y}{2\varepsilon^2}\varepsilon^{n_c}\right) \\ &= b\varepsilon L \ll \left|(b\varepsilon)^y - (b\varepsilon)^x\right|^{-\log_b \varepsilon}. \end{aligned}$$

Moreover, since x and y lie in a bounded interval, $(b\varepsilon)^y - (b\varepsilon)^x \ll |y - x|$. Hence $f(y) - f(x) \ll |y - x|^{-\log_b \varepsilon}$, and so from the periodicity of f it follows that f is Hölder continuous with exponent $-\log_b \varepsilon = \eta$.

In the case that $\eta > 1$, or $\varepsilon < 1/b$, note that the derivation of Equation B.1 still holds. So sending $n \to \infty$ in Equation B.1 yields $|f(y) - f(x)| \leq 0$, or $f(x) = f(y)$. Hence f is constant.

Finally, in the case that $\eta = 1$, or $\varepsilon = 1/b$, note that $h(t)$ is bounded, and so $g(y+n) - g(x+n) \ll |y-x|$. Sending $n \to \infty$, this yields that $f(y) - f(x) \ll 1$, i.e. f is Lipschitz continuous. This finishes the proof. $\square$

Lemma 2.2. *Let $b \in \mathbb{N}\backslash\{1\}$, let $\eta \in (0,1]$ and let $(x_m)_{m\in\mathbb{N}}$ be a sequence of real numbers such that*

(a) $x_m - x_{bm} \ll m^{-\eta}$

(b) $x_m - x_{m-1} \ll m^{-\eta}$

(c) $x_m - (b+1)x_{bm} + bx_{b^2m} \ll m^{-1-\eta}$

(d) $x_m - x_{bm} - x_{m-1} + x_{bm-b} \ll m^{-1-\eta}$

Then there are 1-periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $x_m = f(\log_b m) + \frac{1}{m} g(\log_b m) + O(m^{-1-\eta})$ for large m.

Proof. Define $y_m = m(x_m - x_{bm})$ for $m \in \mathbb{N}$. Then

$$\begin{aligned} y_m - y_{bm} &= m(x_m - x_{bm} - b(x_{bm} - x_{b^2m})) \\ &= m(x_m - (b+1)x_{bm} + bx_{b^2m}) \ll m^{-\eta} \end{aligned}$$

by the third condition above. Also,

$$\begin{aligned} y_m - y_{m-1} &= m(x_m - x_{bm} - x_{m-1} + x_{bm-b}) + x_{m-1} - x_{bm-b} \\ &= O\big(m^{-\eta}\big) + O\big((m-1)^{-\eta}\big) \ll m^{-\eta} \end{aligned}$$

by the first and fourth conditions above. Hence by Lemma 2.1, there is a 1-periodic continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} y_m &= \frac{b-1}{b}\, g(\log_b m) + O\big(m^{-\eta}\big) \\ \iff \quad x_m - x_{bm} &= \frac{1}{m}\frac{b-1}{b}\, g(\log_b m) + O\big(m^{-1-\eta}\big). \end{aligned}$$

The reason for the factor $\frac{b-1}{b}$ will become apparent shortly.

On the other hand, since x_m satisfies the first and second conditions we have from Lemma 2.1 that

$$x_m = f(\log_b m) + O\big(m^{-\eta}\big)$$

for some 1-periodic continuous function $f : \mathbb{R} \to \mathbb{R}$.

Hence

$$\begin{aligned} x_m - f(\log_b m) &= x_m - \lim_{k\to\infty} x_{b^k m} = \sum_{k=0}^{\infty} x_{b^k m} - x_{b^{k+1} m} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{b^k m} \frac{b-1}{b} \; g(\log_b(b^k m)) + O\big((b^k m)^{-1-\eta}\big) \right) \\ &= \frac{b-1}{bm} \sum_{k=0}^{\infty} \frac{1}{b^k} g(\log_b m) + O\left(m^{-1-\eta} \sum_{k=0}^{\infty} \big(b^{-1-\eta}\big)^k \right) \\ &= \frac{1}{m} g(\log_b m) + O\left(\frac{1}{1-b^{-1-\eta}} \; m^{-1-\eta} \right), \end{aligned}$$

and so

$$x_m = f(\log_b m) + \frac{1}{m} g(\log_b m) + O\big(m^{-1-\eta}\big)$$

as desired. $\square$